Linear Transformations and Group Representations

Homework #1 (2022-2023), Questions

Characteristic equations, etc.

Q1. Find the characteristic equation of $R = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$. Find its roots, i.e., the eigenvalues of R.

Q2. Say A is a linear transformation on V, with a full set of distinct eigenvalues $\lambda_1,...,\lambda_m$, and corresponding eigenvectors $v_1,...,v_m$, and B is a linear transformation on W, with a full set of distinct eigenvalues $\mu_1,...,\mu_n$, and eigenvectors $w_1,...,w_n$. We define $A\otimes B$ as a linear transformation in $V\otimes W$ by its action on elementary tensor products $\left(A\otimes B\right)(v\otimes w) = (Av)\otimes(Bw)$, extended by linearity to all of $V\otimes W$.

A. What are the eigenvalues and eigenvectors of $A\otimes B$?

B. What is $tr(A\otimes B)$, in terms of $tr(A)$ and $tr(B)$?

C. Let $A = B$ and $V = W$. What are the eigenvectors and eigenvalues of $sym(A^{\otimes 2})$, i.e., the action of A in $sym(V^{\otimes 2})$? What are the eigenvectors and eigenvalues of $anti(A^{\otimes 2})$?

D. What is $tr(sym(A^{\otimes 2}))$ and $tr(anti(A^{\otimes 2}))$ in terms of $tr(A)$ and $tr(A^2)$?