Linear Transformations and Group Representations

Homework #2 (2022-2023), Answers

Projections and their relationships

See also Q1 and Q3 of 2021-2022 LTGR homework #2

Q1: Commuting projections

A. Given projections P and Q with $PQ = 0$, is $QP = 0$?
Hint: First show that if $\langle x, z \rangle = 0$ for all x, then $z = 0$. Then consider $\langle PQx, y \rangle$.
For the hint: if $\langle x, z \rangle = 0$ for all x, then, for $x = z$, $\langle z, z \rangle = 0$. But since the inner product is positive-definite, this means that $z = 0$.
Now consider $\langle PQx, y \rangle$. This is zero for all x, since we've hypothesized that $PQ = 0$. But also, $\langle PQx, y \rangle = \langle Qx, Py \rangle = \langle x, QPy \rangle$, applying the self-adjoint property of P, and then self the self-adjoint property of Q. So $\langle x, QPy \rangle = 0$ for all x, and therefore $QPy = 0$. Since this holds for all y, $QP = 0$.

B. What is the geometric interpretation of this?
The subspaces that P and Q project into are have only the origin in common, and, every vector in the P-subspace is orthogonal to every vector in the Q-subspace.

C. Given projections P and Q with $PQ = QP$, is PQ a projection?
We need to show that PQ is self-adjoint and that $(PQ)^2 = PQ$.
For self-adjointness: $\langle PQx, y \rangle = \langle Qx, Py \rangle = \langle x, QPy \rangle = \langle x, PQy \rangle$, applying the self-adjoint property of P, and then self the self-adjoint property of Q, and then commutativity of P and Q.
For idempotence:
$(PQ)^2 = PQPQ = P^2Q^2 = PQ$, where the second equality uses commutativity and the third uses idempotence of P and Q.

D. (Converse of C) Given projections P and Q with PQ a projection, is $PQ = QP$?
$\langle PQx, y \rangle = \langle x, PQy \rangle$ because PQ is a projection and therefore self-adjoint. But also, $\langle PQx, y \rangle = \langle Qx, Py \rangle = \langle x, QPy \rangle$, since P and Q are self-adjoint. So, $\langle x, PQy \rangle = \langle x, QPy \rangle$, which means that $\langle x, (PQ - QP)y \rangle = 0$ for any x and y. Using the hint of part A, this means that $(PQ - QP)y = 0$ for all y, i.e., $PQ - QP = 0$.

Note that here, we didn't use idempotency. So we have actually shown that if P, Q, and PQ are self-adjoint, then $PQ = QP$.

E. Given projections P, Q that commute, and $PQ \neq 0$, consider $X = P - PQ$, $Y = Q - PQ$, and $Z = PQ$:
(i) Show that X, Y, and Z are projections. (ii) What is the geometric interpretation?

For (i), Z , see part C. For X and Y : Note that $X = P(I - Q)$, where $I - Q$ is a projection (since Q is a projection), and P commutes with $I - Q$ since P commutes with Q . So part C also demonstrates that X is a projection, with $I - Q$ playing the role of Q . Similarly, part C demonstrates that Y is a projection, with $I - P$ playing the role of P .

For (ii): Noe that X , Y , and Z all commute – and each pairwise product is zero. For example:

$$\begin{aligned} XY &= (P - PQ)(Q - QP) = PQ - PQQ - PQP + PQQP \\ &= PQ - PQ - P^2Q + P^2Q^2 \\ &= PQ - PQ - PQ + PQ \\ &= 0 \end{aligned}$$

. So the ranges of X , Y , and Z are all mutually orthogonal. A slightly different viewpoint: PQ is the intersection of the ranges of P and Q , and $X = P - PQ$, $Y = Q - PQ$ "project out" this intersection.