Linear Transformations and Group Representations

Homework #3 (2022-2023), Answers

Representations of the full symmetric group, and related

Q1: Some irreducible representations of S_n *, the group of all permutations of* n *objects.*
A permutation on n *objects has an action on vectors in a vector space* V *of dimension* n *by permuting its coordinates. This yields a unitary representation* U *of dimension* n*. Here we determine the irreducible components of this representation.*

A. Consider the subspace Y *of* V *of all vectors* $\vec{v}=(v_1,...,v_n)$ *in which the coordinates are equal. How do the permutations act on* Y *? What does this mean about the reducibility of* U *?*

Permutations map all vectors in Y into themselves – so U preserves Y , and hence identifies a one-dimensional subspace of V in which U acts, i.e., an irreducible component.

B. What is the projection onto Y *? What is the projection onto the complementary subspace, here denoted* Z *?*

Consider the average of all of the actions of the group on a typical vector:
$P_Y(\vec{v})=\frac{1}{|G|}\sum_{g\in G}L_g(\vec{v})=\frac{1}{n!}\sum_{\sigma\in S_n}(v_{\sigma(1)},v_{\sigma(2)},...,v_{\sigma(n)})$. As σ ranges over all permutations, each $\sigma(k)$ takes on all values in $\{1,...,n\}$ equally often,i.e., $\frac{1}{n}n!$ times . So each coordinate of $P_Y(\vec{v})$ is the average of all the coordinate of $\vec{v}$: $P_Y(\vec{v})=\left(\frac{1}{n}\sum_{i=1}^{n}v_i\right)\vec{1}$, where $\vec{1}$ is the vector all of whose coordinates are equal to 1.
The projection onto the complementary subspace is $I-P_Y$, which takes $\vec{v}$ to $\vec{v}-P_Y\vec{v}$, i.e., it subtracts the mean value of the coordinates of $\vec{v}$ from each coordinate. For any vector in Z, the sum of the coordinates is zero.

C. Show that (for $n\geq 3$*), the representation* U *is not reducible in* Z*. Hint: First show that if there is any nonzero vector in* Z*, then, by considering some of the group actions, show that* Z *also contains a vector that has all but one of its coordinates identical, and the remaining coordinate equal to a distinct value. Then, by considering other group actions on this vector, show that this vector and its images span* Z*.*

Take a nonzero $\vec{c}\in Z$. Consider all of the permutations σ that keep the first index unchanged, and permute the remaining indices. Averaging $\sigma(\vec{c})$ yields a vector in which all coordinates other than the first have an identical value, i.e., of the form $\sigma(\vec{c})=(c_1,c_0,...,c_0)$. Since $\sigma(\vec{c})$ is in Z (it is constructed as a linear combination of vectors in Z), its coordinates sum to zero, i.e., $c_1=-(n-1)c_0$. This means that
$\vec{b}_1=\frac{(n-1)}{c_1}\vec{c}=(n-1,-1,-1,...,-1)$ is in Z.
Now consider the group actions that move the oddball entry of $\vec{b}_1$ to each of the other entries, producing vectors $\vec{b}_k$, with an $n-1$ in the k th position and -1 in all other positions. The n vectors $\vec{b}_1,...,\vec{b}_n$ cannot be linearly independent, since they all lie in Z, and Z has dimension $n-1$. But if we can show that the $n-1$ vectors $\vec{b}_1,...,\vec{b}_{n-1}$ are linearly independent, then they necessarily span Z (since, otherwise, adjoining a vector outside their span, but inside Z, would lead to a basis of size $\geq n$ for a the $n-1$-dimensional space Z).

To show that the $n-1$ vectors $\vec{b}_1,...,\vec{b}_{n-1}$ are linearly independent: Assume otherwise. Then there are some scalars $a_1,...,a_{n-1}$, not all zero, for which $\vec{s}=\sum_{k=1}^{n-1} a_k\vec{b}_k=0$. The j th coordinate of $\vec{s}$ (for $j=1,...,n-1$)is $s_j=(n-1)a_j+\sum_{k\neq j}(-1)a_k=(n-1)a_j+\sum_{k=1}^{n-1}(-1)a_k+a_j=na_j-\sum_{k=1}^{n-1}a_k$. If $\vec{s}=0$, then all of these quantities are zero, i.e., each $a_j=\frac{1}{n}\sum_{k=1}^{n-1}a_k$. But the n th coordinate of $\vec{s}$, which also must be 0, is $-\sum_{k=1}^{n-1}a_k$. So this means that all of the $a_1,...,a_{n-1}$ are 0 – and that the vectors $\vec{b}_1,...,\vec{b}_{n-1}$ are linearly independent.

D. We now have three irreducible representations of S_n: the trivial representation I that maps all permutations to 1, the parity representation (here, called P) that maps all permutations to ± 1, depending on their parity, and the representation (here, called L) given by restricting U above to the $n-1$-dimensional subspace in which it acts non-trivially. Use the characters to show that $L\otimes P$ is also irreducible.

$L\otimes P$ is irreducible if $\frac{1}{|G|}\sum_{g\in G}\left|\chi_{L\otimes P}(g)\right|^2=1$. But $\chi_{L\otimes P}(g)=\chi_L(g)\chi_P(g)$. So:

$\frac{1}{|G|}\sum_{g\in G}\left|\chi_{L\otimes P}(g)\right|^2=\frac{1}{|G|}\sum_{g\in G}\left|\chi_L(g)\chi_P(g)\right|^2=\frac{1}{|G|}\sum_{g\in G}\left|\chi_L(g)\right|^2\left|\chi_P(g)\right|^2=\frac{1}{|G|}\sum_{g\in G}\left|\chi_L(g)\right|^2$, and the latter is equal to 1 because L is irreducible.

E. Now consider the subgroup of S_n consisting of only the even-parity permutations (known as the alternating group, A_n). Each of the representations of S_n is also a representation of A_n. But which one(s) of the above are distinct? And which one(s) are irreducible?

On restricting to A_n, the parity representation P becomes the trivial representation (since A_n is defined as the subgroup of S_n consisting only of even parity). And therefore $L\otimes P$ becomes identical to L.

But does L remain irreducible? We need to check that the argument in part C goes through. The crucial step is the first one: whether one can permute the entries $2,...,n$ of $\vec{c}$ while keeping the first entry in place. For $n\geq 4$, this is possible – e.g., the permutation (uvw), where u, v, and w are all distinct integers in $\{2,...,n\}$. But not when $n=3$: the only possible permutation is (23), and this is permutation is odd. So, for $n\geq 4$, the representation is irreducible.

For $n=3$, the group A_3 only has three elements: the identity, (123), and (132). It is the same as the cyclic group on three elements, $\mathbb{Z}_3$, and commutative. So (see notes) all of its irreducible representations are one-dimensional. The representation L, which is two-dimensional, must be the direct sum of two of them. Since it does not leave any vector invariant, it is the sum of the two non-trivial one-dimensional representations of A_3 (or $\mathbb{Z}_3$).

Q2: S_5 in detail

Here we find all of the irreducible representations of S_5, the group of all permutations of 5 objects, i.e., we construct its complete character table. The first step is to determine the conjugate classes. Each conjugate class corresponds to a way of partitioning 5 objects into disjoint cycles.

Conjugate class	*ident.*	*(AB)*	*(AB)(CD)*	*(ABC)*	*(ABC)(DE)*	*(ABCD)*	*(ABCDE)*
Size	*1*	*10*	*15*	*20*	*20*	*30*	*24*

(check): total number of elements is $1+10+15+20+20+30+24=120=5!$.

A. Now add the identity representation I and the parity representation P, and check that I and P are orthogonal:

Conjugate class	ident.	(AB)	(AB)(CD)	(ABC)	(ABC)(DE)	(ABCD)	(ABCDE)
Size	1	10	15	20	20	30	24
χ_I	1	1	1	1	1	1	1
χ_P	1	-1	1	1	-1	-1	1

check orthogonality: $\frac{1}{|G|}\sum_{g\in G}\chi_I(g)\overline{\chi_P(g)}=\frac{1}{120}(1-10+15+20-20-30+24)=0$.

B. Consider the unitary representation as permutation matrices, as in Q1. Call it X. Use the characters to show that X is reducible. Find which of the above representations is contained in X and project it out to obtain L

Note that the character of X is the number of objects unchanged by a representative permutation. So:

Conjugate class	ident.	(AB)	(AB)(CD)	(ABC)	(ABC)(DE)	(ABCD)	(ABCDE)
Size	1	10	15	20	20	30	24
χ_X	5	3	1	2	0	1	0

$\frac{1}{|G|}\sum_{g\in G}|\chi_X(g)|^2=\frac{1}{120}(1\bullet 5^2+10\bullet 3^2+15\bullet 1^2+20\bullet 2^2+20\bullet 0^2+30\bullet 1^2+24\bullet 0^2)=\frac{240}{120}=2$, so X has two irreducible components. Clearly $\frac{1}{|G|}\sum_{g\in G}\chi_X(g)\overline{\chi_I(g)}>0$, since neither has any negative entries. So we know that $X=I\oplus L$ where L is irreducible, and we can compute the character of L from $\chi_X(g)=\chi_I(g)+\chi_L(g)$:

Conjugate class	ident.	(AB)	(AB)(CD)	(ABC)	(ABC)(DE)	(ABCD)	(ABCDE)
Size	1	10	15	20	20	30	24
χ_I	1	1	1	1	1	1	1
χ_P	1	-1	1	1	-1	-1	1
χ_L	4	2	0	1	-1	0	-1

C. Compute the character of $L\otimes P$, verify that L and $L\otimes P$ are irreducible, and verify that χ_L and $\chi_{L\otimes P}$ are orthogonal functions on the group.

$\chi_{L\otimes P}(g)=\chi_L(g)\chi_P(g)$, so now we have

Conjugate class	ident.	(AB)	(AB)(CD)	(ABC)	(ABC)(DE)	(ABCD)	(ABCDE)
Size	1	10	15	20	20	30	24
χ_I	1	1	1	1	1	1	1

χ_P	1	-1	1	1	-1	-1	1
χ_L	4	2	0	1	-1	0	-1
$\chi_{L\otimes P}$	4	-2	0	1	1	0	-1

To verify irreducibility of L:

$$\frac{1}{|G|}\sum_{g\in G}\left|\chi_L(g)\right|^2=\frac{1}{120}\left(1\bullet4^2+10\bullet2^2+15\bullet0^2+20\bullet1^2+20\bullet(-1)^2+30\bullet0^2+24\bullet(-1)^2\right)=\frac{120}{120}=1$$

Irreducibility of $L\otimes P$ follows immediately, since $\left|\chi_{L\otimes P}(g)\right|^2=\left|\chi_L(g)\chi_P(g)\right|^2=\left|\chi_L(g)\right|^2$. Note that $\left|\chi_P(g)\right|^2=1$ holds not only for the parity representation, but would also hold for any one-dimensional unitary representation.

To verify orthogonality of χ_L and $\chi_{L\otimes P}$:

$$\frac{1}{|G|}\sum_{g\in G}\chi_L(g)\overline{\chi_{L\otimes P}(g)}=\frac{1}{|G|}\sum_{g\in G}\chi_L(g)\overline{\chi_L(g)\chi_P(g)}=\frac{1}{|G|}\sum_{g\in G}\left|\chi_L(g)\right|^2\chi_P(g)$$
$$=\frac{1}{120}\left(1\bullet4^2-10\bullet2^2+15\bullet0^2+20\bullet1^2-20\bullet(-1)^2-30\bullet0^2+24\bullet(-1)^2\right)=0$$

D. To find another irreducible representation, observe that S_5 also acts on the 10 unordered pairs of letters. For example, the permutation that cycles (BDE) does the following: it takes the pair $\{A,B\}$ to the pair $\{A,D\}$, leaves the pair $\{A,C\}$ unchanged, it takes $\{A,D\}$ to $\{A,E\}$, it takes $\{A,E\}$ to $\{A,B\}$, it takes $\{B,C\}$ to $\{D,C\}$ (which is equivalent to $\{C,D\}$), etc. So S_5 has a representation as permutation matrices of 10 objects (the 10 letter pairs). Call this Y. Determine its character and show that it is reducible.

Conjugate class	ident.	(AB)	(AB)(CD)	(ABC)	(ABC)(DE)	(ABCD)	(ABCDE)
Size	1	10	15	20	20	30	24
χ_I	1	1	1	1	1	1	1
χ_P	1	-1	1	1	-1	-1	1
χ_L	4	2	0	1	-1	0	-1
$\chi_{L\otimes P}$	4	-2	0	1	1	0	-1
χ_Y	10	4	2	1	1	0	0

(For example, the permutation (AB) preserves the unordered pair $\{A,B\}$, and the three unordered pairs that do not contain A or B. So it preserves four un-ordered pairs.)

Character:

$\frac{1}{|G|}\sum_{g\in G}\left|\chi_Y(g)\right|^2=\frac{1}{120}\left(1\bullet10^2+10\bullet4^2+15\bullet2^2+20\bullet1^2+20\bullet1^2+30\bullet0^2+24\bullet0^2\right)=\frac{360}{120}=3$, so Y has three irreducible components.

E. Determine which of the previously-found irreducible representations are components of Y, and project them out to obtain an irreducible representation, M.

$\frac{1}{|G|}\sum_{g\in G}\chi_Y(g)\overline{\chi_I(g)}>0$, since neither has any negative entries. But also,

$\frac{1}{|G|}\sum_{g\in G}\chi_Y(g)\overline{\chi_L(g)}=\frac{1}{120}\left(1\bullet10\bullet4+10\bullet4\bullet2+15\bullet2\bullet0+20\bullet1\bullet1+20\bullet1\bullet(-1)+30\bullet0\bullet0+24\bullet0\bullet(-1)\right)=1$. So that $Y=I\oplus L\oplus M$ where M is irreducible, and $\chi_Y(g)=\chi_I(g)+\chi_L(g)+\chi_M(g)$.

Conjugate class	ident.	(AB)	(AB)(CD)	(ABC)	(ABC)(DE)	(ABCD)	(ABCDE)
Size	1	10	15	20	20	30	24
χ_I	1	1	1	1	1	1	1
χ_P	1	-1	1	1	-1	-1	1
χ_L	4	2	0	1	-1	0	-1
$\chi_{L\otimes P}$	4	-2	0	1	1	0	-1
χ_M	5	1	1	-1	1	-1	0

F. Compute the character of $M\otimes P$, verify that M and $M\otimes P$ are irreducible, and verify that χ_M and $\chi_{M\otimes P}$ are orthogonal functions on the group.

Conjugate class	ident.	(AB)	(AB)(CD)	(ABC)	(ABC)(DE)	(ABCD)	(ABCDE)
Size	1	10	15	20	20	30	24
χ_I	1	1	1	1	1	1	1
χ_P	1	-1	1	1	-1	-1	1
χ_L	4	2	0	1	-1	0	-1
$\chi_{L\otimes P}$	4	-2	0	1	1	0	-1
χ_M	5	1	1	-1	1	-1	0
$\chi_{M\otimes P}$	5	=1	1	-1	-1	1	0

To verify irreducibility of M (and, as per part C, for $M\otimes P$:

$$\frac{1}{|G|}\sum_{g\in G}|\chi_M(g)|^2=\frac{1}{120}\left(1\bullet 5^2+10\bullet 1^2+15\bullet 1^2+20\bullet(-1)^2+20\bullet 1^2+30\bullet(-1)^2+24\bullet 0^2\right)=\frac{120}{120}=1.$$

To verify orthogonality of χ_M *and* $\chi_{M\otimes P}$:

$$\frac{1}{|G|}\sum_{g\in G}\chi_M(g)\overline{\chi_{M\otimes P}(g)}=\frac{1}{|G|}\sum_{g\in G}\chi_M(g)\overline{\chi_M(g)\chi_P(g)}=\frac{1}{|G|}\sum_{g\in G}|\chi_M(g)|^2\chi_P(g)$$
$$=\frac{1}{120}\left(1\bullet 5^2-10\bullet 1^2+15\bullet 1^2+20\bullet(-1)^2-20\bullet 1^2-30\bullet 1^2+24\bullet 0^2\right)=0$$

G. At this point, we have found 6 irreducible representations. There must be a seventh one, N, since there are seven conjugate classes. Determine its dimension, and then complete the character table by using "row orthonormality", i.e. that the characters are orthonormal functions of the group elements.

The sum of the squares of the dimension is the order of the group, and the dimension of N is its character at the identity element. So $120=|G|=\left(1^2+1^2+4^2+4^2+5^2+5^2+\left(\chi_N(ident)\right)^2\right)$, and $\chi_N=6$. The full character table is

Conjugate class	ident.	(AB)	(AB)(CD)	(ABC)	(ABC)(DE)	(ABCD)	(ABCDE)
Size	1	10	15	20	20	30	24
χ_I	1	1	1	1	1	1	1
χ_P	1	-1	1	1	-1	-1	1
χ_L	4	2	0	1	-1	0	-1

$\chi_{L\otimes P}$	4	-2	0	1	1	0	-1
χ_M	5	1	1	-1	1	-1	0
$\chi_{M\otimes P}$	5	-1	1	-1	-1	1	0
χ_N	6	0	-2	0	0	0	1

Q3: A_5 in detail

A_5 is the group of all even-parity permutations of 5 objects. Since it is a subgroup of S_5, all of the irreducible representations of S_5 are also representations of A_5, but some may be reducible. Here, we analyze this situation, and thereby determine the character table of A_5.
The first step is to determine the conjugate classes of A_5. We only need to consider the conjugate classes of S_5 that are even permutations, but we also have to check whether they split – since elements g and h that are conjugate in S_5, i.e., $s^{-1}gs = h$ for some $s \in S_5$ may not be conjugate in A_5.

Conjugate class in S_5	*ident.*	*(AB)(CD)*	*(ABC)*	*(ABCDE)*
Size	*1*	*15*	*20*	*24*

We can conjugate any element of the form $(AB)(CD)$ to any other element by an even permutation, since, if an odd permutation σ suffices, we can find an even permutation that will suffice by using $\tau = \sigma \circ (AB)$.
Similarly, we can conjugate any element of the form (ABC) to any other element by an even permutation, since, if an odd permutation σ suffices, we can find an even permutation that will suffice by using $\tau = \sigma \circ (DE)$.
But we can only conjugate $(ABCDE)$ to other 5-cycles that differ by an even permutation. So that conjugate class splits:

Conjugate class in A_5	*ident.*	*(AB)(CD)*	*(ABC)*	*(ABCDE)*	*(BACDE)*
Size	*1*	*15*	*20*	*12*	*12*

A. For the irreducible representations of S_5 (I, P, L, $L\otimes P$, M, $M\otimes P$, and N), which ones are indistinguishable on A_5, and which ones remain irreducible?

Representations that differ just by tensoring with P are identical A_5, since P becomes the identity representation on A_5. This leaves us with representations that all have different dimensions, which remain distinct: I, L, M, and N.

So far, we have this work-in-progress character table:

Conjugate class in A_5	ident.	(AB)(CD)	(ABC)	(ABCDE)	(BACDE)
Size	1	15	20	12	12
χ_I	1	1	1	1	1
χ_L	4	0	1	-1	-1
χ_M	5	1	-1	0	0

χ_N	6	-2	0	1	1

It follows from the trace formula that I, L, M are irreducible. But for N,
$\frac{1}{|G|}\sum_{g\in G}|\chi_N(g)|^2 = \frac{1}{60}\left(1\bullet 6^2 + 15\bullet(-2)^2 + 20\bullet 0^2 + 12\bullet 1^2 + 12\bullet 1^2\right) = \frac{120}{60} = 2$, so N has two irreducible components.

B. Use row orthonormality to complete the character table.

The dimensions of the two components of N must satisfy $60 = 1^2 + 4^2 + 5^2 + d_1^2 + d_2^2$ (and, $d_1 + d_2 = 6$), so they are both of dimension 3. Their characters sum to χ_N. The characters also must be real, as each group element and its inverse is in the same conjugate class. (For example, the inverse of (ABCDE) is (AEDCB), which differs from (ABCDE) by conjugation with (BE)(CD). Also, the last two columns of the character table must be symmetric, since conjugation by (AB) is an automorphism of the group (i.e., an isomorphism from the group to itself).

Conjugate class in A_5	ident.	(AB)(CD)	(ABC)	(ABCDE)	(BACDE)
Size	1	15	20	12	12
χ_I	1	1	1	1	1
χ_L	4	0	1	-1	-1
χ_M	5	1	-1	0	0
$\chi_{N_\backslash}$	3	-1	0	ξ	$1-\xi$
$\chi_{N_/}$	3	-1	0	$1-\xi$	ξ

This, along with row orthonormality, allows us to complete the character table:
$\frac{1}{|G|}\sum_{g\in G}\chi_{N_\backslash}(g)\overline{\chi_{N_/}(g)} = \frac{1}{60}\left(1\bullet 3^2 + 15\bullet(-1)^2 + 20\bullet 0^2 + 12\bullet\xi(1-\xi) + 12\bullet(1-\xi)\xi\right) = 0$, so
$24 + 12\bullet\xi(1-\xi) + 12\bullet(1-\xi)\xi = 0 \Leftrightarrow$

$\xi(1-\xi) + 1 = 0 \Leftrightarrow \xi^2 - \xi - 1 = 0 \Leftrightarrow \xi = \frac{1\pm\sqrt{5}}{2}$.

This result makes more sense once one recognizes that $\xi = 1 + \theta + \theta^4$ and $1 - \xi = 1 + \theta^2 + \theta^3$ for $\theta = e^{\frac{2\pi i}{5}}$. (One way to show this is to see that they satisty the above quadratic equation). The meaning of this is that $\chi_{N_\backslash}$ and $\chi_{N_/}$ are representations of A_5 as three-dimensional rotations, and (ABCDE) is a rotation by 1/5 of a circle. There's a coloring of the edges of the icosahedron with five colors, for which every member of A_5 corresponds to a rotation of the icosahedron.