Linear Transformations and Group Representations

Homework #3 (2022-2023), Questions

Representations of the full symmetric group, and related

Q1: Some irreducible representations of S_n, the group of all permutations of n objects.
A permutation on n objects has an action on vectors in a vector space V of dimension n by permuting its coordinates. This yields a unitary representation U of dimension n. Here we determine the irreducible components of this representation.

A. Consider the subspace Y of V of all vectors $\vec{v} = (v_1, ..., v_n)$ in which the coordinates are equal. How do the permutations act on Y?What does this mean about the reducibility of U?

B. What is the projection onto Y?What is the projection onto the complementary subspace, here denoted Z ?

C. Show that (for $n \geq 3$), the representation U is not reducible in Z. Hint: First show that if there is any nonzero vector in Z, then, by considering some of the group actions, show that Z also contains a vector that has all but one of its coordinates identical, and the remaining coordinate equal to a distinct value. Then, by considering other group actions on this vector, show that this vector and its images span Z.

D. We now have three irreducible representations of S_n: the trivial representation I that maps all permutations to 1, the parity representation (here, called P) that maps all permutations to ± 1, depending on their parity, and the representation (here, called L) given by restricting U above to the $n-1$-dimensional subspace in which it acts non-trivially. Use the characters to show that $L \otimes P$ is also irreducible.

Q2: S_5 in detail

Here we find all of the irreducible representations of S_5, the group of all permutations of 5 objects, i.e., we construct its complete character table. The first step is to determine the conjugate classes. Each conjugate class corresponds to a way of partitioning 5 objects into disjoint cycles.

Conjugate class	ident.	(AB)	(AB)(CD)	(ABC)	(ABC)(DE)	(ABCD)	(ABCDE)
Size	1	10	15	20	20	30	24

(check): total number of elements is $1+10+15+20+20+30+24=120=5!$.

A. Now add the identity representation I and the parity representation P, and check that I and P are orthogonal:

B. Consider the unitary representation as permutation matrices, as in Q1. Call it X. Use the characters to show that X is reducible. Find which of the above representations is contained in X and project it out to obtain L

C. Compute the character of $L \otimes P$, verify that L and $L \otimes P$ are irreducible, and verify that χ_L and $\chi_{L \otimes P}$ are orthogonal functions on the group.

D. To find another irreducible representation, observe that S_5 also acts on the 10 unordered pairs of letters. For example, the permutation that cycles (BDE) does the following: it takes the pair $\{A,B\}$ to the pair $\{A,D\}$, leaves the pair $\{A,C\}$ unchanged, it takes $\{A,D\}$ to $\{A,E\}$, it takes $\{A,E\}$ to $\{A,B\}$, it takes $\{B,C\}$ to $\{D,C\}$ (which is equivalent to $\{C,D\}$), etc. So S_5 has a representation as permutation matrices of 10 objects (the 10 letter pairs). Call this Y. Determine its character and show that it is reducible.

E. Determine which of the previously-found irreducible representations are components of Y, and project them out to obtain an irreducible representation, M.

F. Compute the character of $M \otimes P$, verify that M and $M \otimes P$ are irreducible, and verify that χ_M and $\chi_{M \otimes P}$ are orthogonal functions on the group.

G. At this point, we have found 6 irreducible representations. There must be a seventh one, N, since there are seven conjugate classes. Determine its dimension, and then complete the character table by using "row orthonormality", i.e. that the characters are orthonormal functions of the group elements.

Q3: A_5 in detail

A_5 is the group of all even-parity permutations of 5 objects. Since it is a subgroup of S_5, all of the irreducible representations of S_5 are also representations of A_5, but some may be reducible. Here, we analyze this situation, and thereby determine the character table of A_5.

The first step is to determine the conjugate classes of A_5. We only need to consider the conjugate classes of S_5 that are even permutations, but we also have to check whether they split – since elements g and h that are conjugate in S_5, i.e., $s^{-1}gs = h$ for some $s \in S_5$ may not be conjugate in A_5.

Conjugate class in S_5	ident.	(AB)(CD)	(ABC)	(ABCDE)
Size	1	15	20	24

We can conjugate any element of the form $(AB)(CD)$ to any other element by an even permutation, since, if an odd permutation σ suffices, we can find an even permutation that will suffice by using $\tau = \sigma \circ (AB)$.
Similarly, we can conjugate any element of the form (ABC) to any other element by an even permutation, since, if an odd permutation σ suffices, we can find an even permutation that will suffice by using $\tau = \sigma \circ (DE)$.
But we can only conjugate $(ABCDE)$ to other 5-cycles that differ by an even permutation. So that conjugate class splits:

Conjugate class in A_5	ident.	(AB)(CD)	(ABC)	(ABCDE)	(BACDE)
Size	1	15	20	12	12

A. For the irreducible representations of S_5 (I, P, L, $L \otimes P$, M, $M \otimes P$, and N), which ones are indistinguishable on A_5, and which ones remain irreducible?
B. Use row orthonormality to complete the character table.