Groups, Fields, and Vector Spaces

Homework #3 (2024-2025), Questions

Q1: Duals of infinite-dimensional spaces

A. Consider the (infinite-dimensional) vector space V of real-valued functions $f(x)$ on the interval $[0,1]$ that are "nice" – continuous, smooth, integrable. For any $g \in V$, there is a mapping Ag from V to the base field, defined by $\left(Ag\right)(f) = \int_0^1 g(x)f(x)dx$. Show that $Ag \in V^*$, i.e., that it is a linear map from V to the base field.

B. For any $y \in [0,1]$, there is another map from V to the base field, defined by $\left(By\right)(f) = f(y)$. Show that $By \in V^*$, i.e., that it is a linear map from V to the base field.

C. For any $y \in [0,1]$, is there a $g \in V$ for which $By = Ag$?

Q2: Direct path to the trace as an intrinsic property of vector-space homomorphisms (AKA matrices)

This homework is closely modeled after a Quora posting of Senia Sheydvasser. Assistant Professor Department of Mathematics, Bates College

Consider a vector space V and its dual space V^*. Elements in $V^* \otimes V$ are sums of elementary tensor products $\Phi = \phi \otimes v$, for $x \in V$ and $\phi \in V^*$. Thus, elements in $V^* \otimes V$ can also be considered to be in $Hom(V,V)$ (i.e., there is a natural homomorphism L from $V^* \otimes V$ to $Hom(V,V)$). The correspondence L between $V^* \otimes V$ to $Hom(V,V)$ takes an elementary tensor product $\Phi = \phi \otimes v$ to $L(\Phi) \in Hom(V,V)$ given by $L(\Phi)(w) = \phi(w)v$. (Note $\phi(w)$ is a scalar). L is then extended to sums of elementary tensor products by linearity.

A. For $\Phi_1 = \phi_1 \otimes v_1$ and $\Phi_2 = \phi_2 \otimes v_2$, determine the action of $L(\Phi_1) \circ L(\Phi_2)$ on an arbitrary $w \in V$, where $\circ$ is composition in $Hom(V,V)$. Express this as the image under L of an elementary tensor product $\Phi_{12} \in V^* \otimes V$. Use this to define a composition rule in $V^* \otimes V$, $\Phi_{12} = \Phi_1 \circ \Phi_2$, for which $L(\Phi_{12}) = L(\Phi_1) \circ L(\Phi_2)$

B. Determine the $\Phi_{21} \in V^* \otimes V$ for which $L(\Phi_{21}) = L(\Phi_2) \circ L(\Phi_1)$.

C. There is also a natural mapping T from $V^* \otimes V$ to the base field of scalars, defined by $T(\Phi) = \phi(v)$ for elementary tensor products and extended to all of $V^* \otimes V$ by linearity. Determine $T(\Phi_{12})$ and $T(\Phi_{21})$. What happens?

D. Now interpret T in coordinates. Specifically, take the "one-hot" basis for V, $v_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, v_2 = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \cdots, v_n = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}$,

and a typical vector in $x \in V$, $x = \sum_{k=1}^{n} x_k v_k$. Take the one-hot basis for V^*, where ϕ_k maps $\phi_k(v_k) = 1$ but

$\phi_j(v_k) = 0$ for $j \neq k$, and a typical $\varphi \in V^*$, $\varphi = \sum_{k=1}^{n} \varphi_k \phi_k$. Written more compactly, $\phi_j(v_k) = \delta_{j,k}$. Then the $M_{j,k} = \phi_j \otimes v_k$ are a basis for $V^* \otimes V$, and an arbitrary $M \in V^* \otimes V$ can be written as $M = \sum_{j,k=1}^{n} m_{j,k} M_{j,k}$, where the $m_{j,k}$ are the matrix entries for M. Determine $T(M_{j,k})$ and then $T(M)$.