Information Theory and Data Analysis

Homework #2 (2024-2025), Answers

Q1: Correlations implied by maximum-entropy

Consider three variables, x_1, x_2, and x_3, with mean zero and unit variance, and with correlations $\langle x_1 x_2 \rangle = a$ and $\langle x_2 x_3 \rangle = b$. In the maximum-entropy distribution that satisfies these constraints, what is $c = \langle x_1 x_3 \rangle$?

The maximum-entropy distribution (notes, and last week's homework) is of the form $p_V(\vec{x}) = \frac{1}{(2\pi)^{n/2}\sqrt{\det V}} \exp\left(-\frac{\vec{x}^T V^{-1}\vec{x}}{2}\right)$, where the nonzero elements of V^{-1} correspond to the constraints on the variances and covariances. That is, V^{-1} is a symmetric matrix of the form $\begin{pmatrix} * & * & 0 \\ * & * & * \\ 0 & * & * \end{pmatrix}$, with the nonzero elements chosen to satisfy the constraints. Say $V = \begin{pmatrix} 1 & a & c \\ a & 1 & b \\ c & b & 1 \end{pmatrix}$. Calculating the inverse of V using minors, the upper right element of V^{-1} -- which must be zero since $\langle x_1 x_3 \rangle$ is unconstrained -- is given by $\det\begin{pmatrix} a & 1 \\ c & b \end{pmatrix} \Big/ \det V$.

So $\det\begin{pmatrix} a & 1 \\ c & b \end{pmatrix} = 0$, i.e., $ab - c = 0$ and $c = ab$.

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