Linear Systems: Black Boxes and Beyond

Homework #1 (2024-2025), Questions

Q1. Some important transfer functions

A. The "boxcar",which averages a signal $s(t)$ over a previous interval τ:
$f(t)=\begin{cases} \frac{1}{\tau}, 0 \le t \le \tau \\ 0 \text{ , } otherwise \end{cases}$. Compute the transfer function, $\hat{f}(\omega)=\int_{-\infty}^{\infty} e^{-i\omega t} f(t)dt$.

B. The delay, i.e., a filter for which the response to a signal $s(t)$ is $r(t)=s(t-\tau)$, the impulse response is $f(t)=\delta(t-\tau)$. Compute the transfer function.

C. Non-causal boxcar averaging, i.e., averaging a signal $s(t)$ over the interval from $-\tau/2$ to $+\tau/2$. Compute the transfer function.

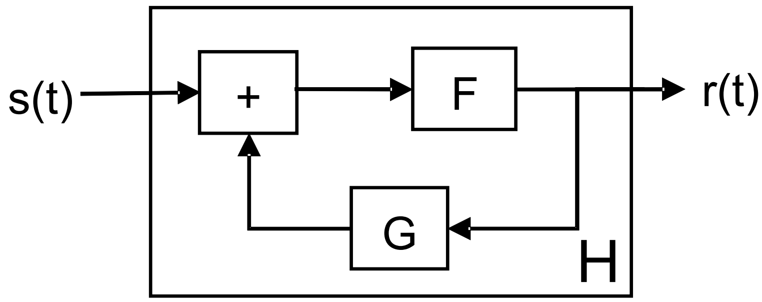
D. The derivative, method 1: Consider a filter f whose output is the time-derivative of the input. First, for any signal $s(t)$. $s'(t)=\lim_{\tau\to 0}\frac{s(t)-s(t-\tau)}{\tau}$. Say f_τ yields $\frac{s(t)-s(t-\tau)}{\tau}$. Using part B, determine $\hat{f}_\tau(\omega)$ and then $\hat{f}(\omega)=\lim_{\tau\to 0}\hat{f}_\tau(\omega)$.

E. The derivative, method 2: If $r(t)=s'(t)$, $\hat{r}(\omega)$ can be directly determined from $\hat{s}(\omega)$, by expressing $s(t)$ in terms of $\hat{s}(\omega)$ and then differentiating.

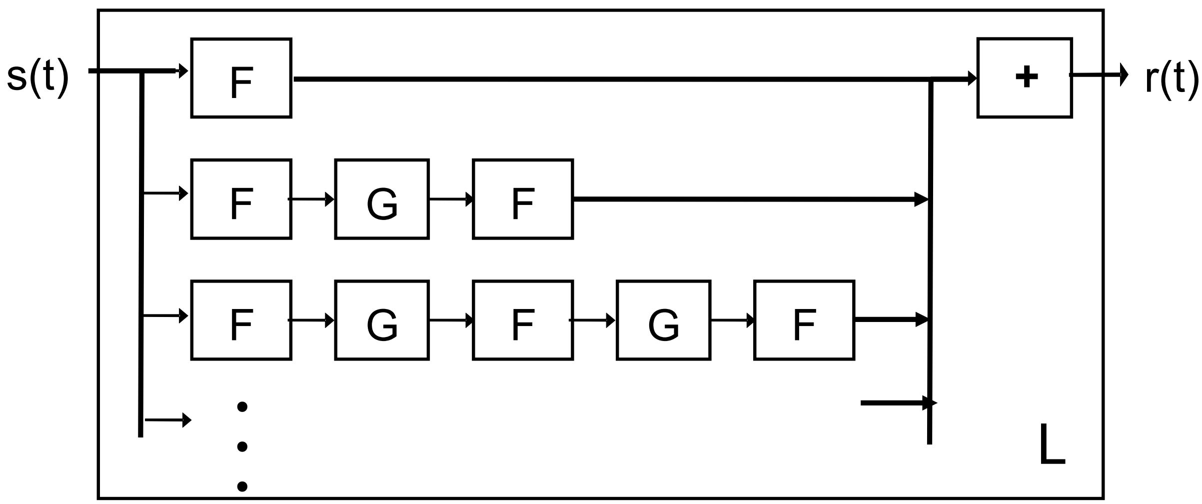
F. From either D or E, what is the transfer function $\hat{f}_n(\omega)$ corresponding to the n th derivative?
$\hat{f}_n(\omega)=\left(\hat{f}_1(\omega)\right)^n$, where $\hat{f}_1(\omega)$ is the first-derivative transfer function of part D or E. So $\hat{f}_n(\omega)=\left(\hat{f}_1(\omega)\right)^n=\left(i\omega\right)^n$.

Q2. Feedback and feedforward

We had determined the transfer function of the composite system H diagrammed here (worked out in class with the feedback signal multiplied by an arbitrary amount k; here, for simplicity, with $k=1$). For this system, $\hat{h}(\omega)=\frac{\hat{f}(\omega)}{1-\hat{f}(\omega)\hat{g}(\omega)}$.



Now, consider the following system, of parallel feedforward elements:



What is its transfer function, $\hat{l}(\omega)$? How does it compare to $\hat{h}(\omega)$?

Q3. The Fourier transform of a Gaussian.

We evaluate $J(D,u)=\int_{-\infty}^{\infty} e^{-\omega^2 D/2} e^{i\omega u} d\omega$.

A. First, consider $I(V)=\int_{-\infty}^{\infty} e^{-x^2/2V} dx$, the integral of a non-normalized Gaussian. Note that I^2 can be considered a two-dimensional integral (say, in x and y), and also an integral in polar coordinates with $r^2=x^2+y^2$. In polar coordinates, the integral is straightforward. This yields I^2 and hence I.

$$I^2=\int_{-\infty}^{\infty} e^{-x^2/2V} dx \int_{-\infty}^{\infty} e^{-y^2/2V} dy = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2/2V} e^{-y^2/2V} dxdy = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)/2V} dxdy .$$

Changing to polar coordinates, with $dxdy = rdrd\theta$:

$I^2=\int_0^{\infty}\int_0^{2\pi} e^{-r^2/2V} rdrd\theta = 2\pi \int_0^{\infty} e^{-r^2/2V} rdr$. With $t=r^2/2$, $dt=rdr$, and

$$I^2 = 2\pi \int_0^{\infty} e^{-t/V} dt = -2\pi V \left(e^{-t/V} \right)\Big|_0^{\infty} = 2\pi V \text{ . So } I = \sqrt{2\pi V} \text{ .}$$

B. Evaluate $\int_{-\infty}^{\infty} e^{-\omega^2 D/2} e^{i\omega u} d\omega$ by completing the square in the exponent.