Linear Transformations and Group Representations

Homework #1 (2024-2025), Answers

Q1: Characteristic equations, eigenvalues, eigenvectors
For each of the following: write the characteristic equation, find the eigenvalues, and find the eigenvectors. Determine if the operator is "normal" (i.e., commutes with its adjoint).

A. $A=\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}$.

Characteristic equation is $\det(zI-A)=0$, i.e., $\det\begin{pmatrix}z & -1\\ 0 & z\end{pmatrix}=0$, i.e., $z^2=0$. Only possible eigenvalue is therefore 0. $Av=A\begin{pmatrix}v_1\\ v_2\end{pmatrix}=\begin{pmatrix}v_2\\ 0\end{pmatrix}$. So if $v=\begin{pmatrix}v_1\\ v_2\end{pmatrix}$ has eigenvalue 0, then $v_2=0$, i.e,. the only eigenvector is $\begin{pmatrix}v_1\\ 0\end{pmatrix}$, which does have eigenvalue 0. A is not normal, as $AA^*=\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}=\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}$ but $A^*A=\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}=\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}$. A does not have a full set of eigenvectors.

B. $B=\begin{pmatrix}1 & -1\\ 0 & 0\end{pmatrix}$.

Characteristic equation is $\det(zI-B)=0$, i.e., $\det\begin{pmatrix}z-1 & 1\\ 0 & z\end{pmatrix}=0$, i.e., $z(z-1)=0$. Possible eigenvalues are therefore 0 and 1. $Bv=B\begin{pmatrix}v_1\\ v_2\end{pmatrix}=\begin{pmatrix}v_1-v_2\\ 0\end{pmatrix}$. So if $v=\begin{pmatrix}v_1\\ v_2\end{pmatrix}$ has eigenvalue 0, then $v_1=v_2$, so one eigenvector is $\begin{pmatrix}v_1\\ v_1\end{pmatrix}$, which does have eigenvalue 0. If $v=\begin{pmatrix}v_1\\ v_2\end{pmatrix}$ has eigenvalue 1, then $v_1=v_1-v_2$, which implies $v_2=0$. $\begin{pmatrix}v_1\\ 0\end{pmatrix}$ does have eigenvalue 1. B is not normal: $BB^*=\begin{pmatrix}1 & -1\\ 0 & 0\end{pmatrix}\begin{pmatrix}1 & 0\\ -1 & 0\end{pmatrix}=\begin{pmatrix}2 & 0\\ 0 & 0\end{pmatrix}$

but

$B^*B=\begin{pmatrix}1 & 0\\ -1 & 0\end{pmatrix}\begin{pmatrix}1 & -1\\ 0 & 0\end{pmatrix}=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$. B has a full set of eigenvectors but they are not orthogonal.

C. $C=\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}$.

Characteristic equation is $\det(zI-C)=0$, i.e., $\det\begin{pmatrix}z-1 & 0\\ 0 & z\end{pmatrix}=0$, i.e., $z(z-1)=0$. Possible eigenvalues are therefore 0 and 1. $Cv=C\begin{pmatrix}v_1\\ v_2\end{pmatrix}=\begin{pmatrix}v_1\\ 0\end{pmatrix}$. So if $v=\begin{pmatrix}v_1\\ v_2\end{pmatrix}$ has eigenvalue 0, then $v_1=0$, so one eigenvector is $\begin{pmatrix}0\\ v_2\end{pmatrix}$, which does have eigenvalue 0. If $v=\begin{pmatrix}v_1\\ v_2\end{pmatrix}$ has eigenvalue 1, then $v_2=0$. $\begin{pmatrix}v_1\\ 0\end{pmatrix}$ does have eigenvalue 1. C is normal, as $C=C^*$; C's eigenvectors form an orthogonal basis.

D. $D=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$.

Characteristic equation is $\det(zI-D)=0$, i.e., $\det\begin{pmatrix} z & -1 \\ 1 & z \end{pmatrix}=0$, i.e., $z^2+1=0$. Possible eigenvalues are therefore $\pm i$. $Dv=D\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}=\begin{pmatrix} v_2 \\ -v_1 \end{pmatrix}$. So if $v=\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ has eigenvalue i, i.e., $\begin{pmatrix} v_2 \\ -v_1 \end{pmatrix}=i\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ then $v_2=iv_1$ and $-v_1=iv_2$, so one eigenvector is $\begin{pmatrix} v_1 \\ iv_1 \end{pmatrix}$, which does have eigenvalue i. Similarly, $\begin{pmatrix} v_1 \\ -iv_1 \end{pmatrix}$ has eigenvalue $-i$. Note that $\begin{pmatrix} v_1 \\ iv_1 \end{pmatrix}$ and $\begin{pmatrix} v_1 \\ -iv_1 \end{pmatrix}$ are orthogonal. D is normal, as $D=-D^*$; D's eigenvectors form an orthogonal basis.

Q2: Tensor Products and Traces (similar to LTGR2223aHW, Q1)

Given a linear transformation A on a vector space V of dimension n, and a complete set of eigenvectors v_i and corresponding eigenvalues λ_i:

A. What are the eigenvectors and eigenvalues of $A\otimes A$?

We can build n^2 distinct eigenvectors from elementary tensor products of the eigenvectors v_i in V: $(A\otimes A)(v_i\otimes v_j)=(Av_i)\otimes(Av_j)=(\lambda_i v_i)\otimes(\lambda_j v_j)=(\lambda_i\lambda_j)(v_i\otimes v_j)$. Since n^2 is the dimension of $A\otimes A$, this is all of the eigenvectors. The eigenvalues are $\lambda_i\lambda_j$ (note that $\lambda_i\lambda_j$ and $\lambda_j\lambda_i$ are counted separately, since $v_i\otimes v_j$ and $v_j\otimes v_i$ are distinct.

B. What are the eigenvectors and eigenvalues of $sym(A\otimes A)$, the restriction of $A\otimes A$ to the symmetric part of $V\otimes V$?

A basis for $sym(V\otimes V)$ are the symmetrized tensor products $v_i\otimes v_j+v_j\otimes v_i$ ($i<j$) and $v_i\otimes v_i$. These are all eigenvectors, with eigenvalues $\lambda_i\lambda_j$ ($\lambda_j\lambda_i$ not counted separately) and λ_i^2.

C. What are the eigenvectors and eigenvalues of $anti(A\otimes A)$, the restriction of $A\otimes A$ to the antisymmetric part of $V\otimes V$?

A basis for $anti(V\otimes V)$ are the antisymmetrized tensor products $v_i\otimes v_j-v_j\otimes v_i$ ($i<j$). These are all eigenvectors, with eigenvalues $\lambda_i\lambda_j$ $i<j$.

D. What is $tr(A\otimes A)$, $tr(sym(A\otimes A))$, and $tr(anti(A\otimes A))$, in terms of $tr(A)$ and $tr(A^2)$?

Since the trace is the sum of the eigenvalues:

$$tr(A\otimes A)=\sum_{i=1}^{n}\sum_{j=1}^{n}\lambda_i\lambda_j=\left(\sum_{i=1}^{n}\lambda_i\right)^2=(trA)^2.$$

For $tr(sym(A\otimes A))$: From part B, $tr(sym(A\otimes A))=\sum_{i<j}^{n}\lambda_i\lambda_j+\sum_{i=1}^{n}\lambda_i^2$. Writing it in a more symmetric form:

$$\sum_{i<j}^{n}\lambda_i\lambda_j+\sum_{i=1}^{n}\lambda_i^2=\frac{1}{2}\sum_{i\neq j}^{n}\lambda_i\lambda_j+\frac{1}{2}\sum_{i=1}^{n}\lambda_i^2+\frac{1}{2}\sum_{i=1}^{n}\lambda_i^2$$

$$=\frac{1}{2}\left(\sum_{i=1}^{n}\sum_{j=1}^{n}\lambda_i\lambda_j\right)+\frac{1}{2}\sum_{i=1}^{n}\lambda_i^2=\frac{1}{2}\left(\sum_{i=1}^{n}\lambda_i\right)\left(\sum_{i=1}^{n}\lambda_i\right)+\frac{1}{2}\sum_{i=1}^{n}\lambda_i^2=\frac{1}{2}(trA)^2+\frac{1}{2}\sum_{i=1}^{n}\lambda_i^2$$

So we need $\sum_{i=1}^{n} \lambda_i^2$. This is the sum of the eigenvalues of A^2. So $tr\left(sym\left(A \otimes A\right)\right) = \frac{1}{2}\left(trA\right)^2 + \frac{1}{2}\left(trA^2\right)$.

For $tr\left(anti(A \otimes A)\right)$: From part C, $tr\left(anti(A \otimes A)\right) = \sum_{i<j}^{n} \lambda_i \lambda_j = tr\left(sym(A \otimes A)\right) - \sum_{i=1}^{n} \lambda_i^2$. So

$tr\left(anti\left(A \otimes A\right)\right) = \frac{1}{2}\left(trA\right)^2 - \frac{1}{2}\left(trA^2\right)$.

Q3: Projections (similar to LTGR2223bHW, Q1)

Given projections P and Q on a vector space V:

A. *Show that if P and Q commute, that PQ is also a projection. What is a geometric interpretation?*

We need to show that PQ is self-adjoint and that $\left(PQ\right)^2 = PQ$.
For the adjoint property:
$(PQ)^* = Q^* P^* = QP = PQ$. (Adjoint of product is product of adjoints in reverse order; P and Q each self-adjoint since they are projections; P and Q commute.)
For idempotency:
$\left(PQ\right)^2 = PQPQ = P^2 Q^2 = PQ$. (Carrying out the multiplication; P and Q commute; P and Q are each idempotent since they are projections.)

The geometric interpretation is that PQ is a projection onto the subspace that is the intersection of the range of P and the range of Q. Note that $P(PQ) = P^2 Q = PQ$, so the range of PQ is in the range of P, and $Q(PQ) = QPQ = PQ^2 = PQ$ so the range of PQ is also in the range of Q.

B. *Show that if P and Q are projections but do not commute, then PQ is not a projection.*

We will show that PQ is not self-adjoint. If it were, then $PQ = (PQ)^*$, which implies that $PQ = Q^* P^*$ (adjoint of product is product of adjoints in reverse order), which implies that $PQ = QP$, since P and Q are projections, implying that (contrary to the hypothesis) that P and Q commute.

C. *If P and Q commute, is $P+Q$ a projection? If not, give a condition on P and Q that guarantees that it is a projection. What is a geometric interpretation?*

Not typically. Take $P = Q$. Then $(P+Q)^2 = \left(2P\right)^2 = 4P^2 = 4P \neq (P+Q)$.
$P+Q$ is self-adjoint $(P+Q)^* = P^* + Q^* = P+Q$. So what we need is that $(P+Q)^2 = P+Q$.
$(P+Q)^2 = (P+Q)(P+Q) = P^2 + PQ + QP + Q^2 = P + PQ + QP + Q = P + 2PQ + Q$, using idempotency for P and Q, and (last step) that P and Q commute.
So, if $(P+Q)^2 = P+Q$, then we must have $PQ = 0$. Conversely, if P and Q are projections with $PQ = QP = 0$, then $P+Q$ is a projection. Geometrically, $P+Q$ is a projection onto the linear span of the range of P and the range of Q.

D. *If P and Q commute, is $P+Q-PQ$ a projection? What is a geometric interpretation?*

Consider $R = I - (P+Q-PQ)$. Note that $R = (I-P)(I-Q)$. If P and Q are projections, then so are $I-P$ and $I-Q$. They commute. By part B, this implies that R is a projection. If R is a projection, then so is $I - R = P + Q - PQ$.

Geometry: $P+Q-PQ$ is a projection onto the linear span of P and Q:
$(P+Q-PQ)P = P^2+QP-PQP = P+QP-QP^2 = P$ and similarly $(P+Q-PQ)Q = Q$, so any vector in the range of either P and Q is in the range of $P+Q-PQ$.

Parts B and D, along with the complementary projection $I-P$, yield a correspondence between Boolean logic (and, or, not) and algebraic operations among a system of commuting projections.

Q4: Inner products in a tensor-product space

Here we show how inner products on a pair of vector spaces can be extended to their tensor product, filling some gaps in the notes. Say the v are vectors in a Hilbert space V with inner product $\langle v,v'\rangle_V$ and similarly the w are vectors in a Hilbert space W with inner product $\langle w,w'\rangle_W$.

A. *Give a natural definition for an inner product $\langle\ ,\ \rangle_{V\otimes W}$ on vectors in $V\otimes W$. Show self-consistency.*

Put $\langle v\otimes w, v'\otimes w'\rangle_{V\otimes W} = \langle v,v'\rangle_V \langle w,w'\rangle_W$ and extend by linearity to sums of elementary tensor products.

Self-consistency: we need $\langle \lambda v\otimes w, v'\otimes w'\rangle_{V\otimes W} = \langle v\otimes \lambda w, v'\otimes w'\rangle_{V\otimes W}$, since $\lambda v\otimes w$ and $v\otimes \lambda w$ are the same elements of $V\otimes W$. (We also need $\langle v\otimes w, \lambda v'\otimes w'\rangle_{V\otimes W} = \langle v\otimes w, v'\otimes \lambda w'\rangle_{V\otimes W}$ but the demonstration is parallel.)
$\langle \lambda v\otimes w, v'\otimes w'\rangle_{V\otimes W} = \langle \lambda v,v'\rangle_V \langle w,w'\rangle_W = \lambda\langle v,v'\rangle_V \langle w,w'\rangle_W$ (definition of $\langle\ ,\ \rangle_{V\otimes W}$, linearity of $\langle\ ,\ \rangle_V$)
But similarly,
$\langle v\otimes \lambda w, v'\otimes w'\rangle_{V\otimes W} = \langle v,v'\rangle_V \langle \lambda w,w'\rangle_W = \lambda\langle v,v'\rangle_V \langle w,w'\rangle_W$ (definition of $\langle\ ,\ \rangle_{V\otimes W}$, linearity of $\langle\ ,\ \rangle_W$)

B. *Show that the properties of an inner product (linearity, conjugate symmetry, and positive-definiteness) hold.*

We need linearity of scalar multiplication and addition. For scalar multiplication:
$\langle \lambda(v\otimes w), v'\otimes w'\rangle_{V\otimes W} = \lambda\langle v\otimes w, v'\otimes w'\rangle_{V\otimes W}$. To show this:
$\langle \lambda(v\otimes w), v'\otimes w'\rangle_{V\otimes W} = \langle \lambda v\otimes w, v'\otimes w'\rangle_{V\otimes W}$ (scalar multiplication of an elementary tensor product)
$\langle \lambda v\otimes w, v'\otimes w'\rangle_{V\otimes W} = \langle \lambda v,v'\rangle_V \langle w,w'\rangle_W$ (definition of $\langle\ ,\ \rangle_{V\otimes W}$)
$\langle \lambda v,v'\rangle_V \langle w,w'\rangle_W = \lambda\langle v,v'\rangle_V \langle w,w'\rangle_W$ (linearity of $\langle\ ,\ \rangle_V$)
$\lambda\langle v,v'\rangle_V \langle w,w'\rangle_W = \lambda\langle v\otimes w, v'\otimes w'\rangle_{V\otimes W}$ (definition of $\langle\ ,\ \rangle_{V\otimes W}$)

$\langle a(v_1\otimes w_1)+b(v_2\otimes w_2), v'\otimes w'\rangle_{V\otimes W} = a\langle v_1\otimes w_1, v'\otimes w'\rangle_{V\otimes W} + b\langle v_2\otimes w_2, v'\otimes w'\rangle_{V\otimes W}$.
For addition, linearity follows because the inner product is extended by linearity from elementary tensor products.

Conjugate symmetry, $\langle v\otimes w, v'\otimes w'\rangle_{V\otimes W} = \overline{\langle v'\otimes w', v\otimes w\rangle}_{V\otimes W}$
First,
$\langle v\otimes w, v'\otimes w'\rangle_{V\otimes W} = \langle v,v'\rangle_V \langle w,w'\rangle_W = \overline{\langle v',v\rangle}_V \overline{\langle w',w\rangle}_W = \overline{\langle v',v\rangle_V \langle w',w\rangle_W} = \overline{\langle v'\otimes w', v\otimes w\rangle}_{V\otimes W}$

(definition of $\langle\ ,\ \rangle_{V\otimes W}$; conjugate symmetry for $\langle\ ,\ \rangle_V$ and $\langle\ ,\ \rangle_W$; properties of complex-conjugation; definition $\langle\ ,\ \rangle_{V\otimes W}$)

Positive definiteness:
Say the $\{v_i\}$ are an orthonormal basis for V and the $\{w_j\}$ are an orthonormal basis for W. Then $\{v_i\otimes w_j\}$ are an orthonormal basis for $V\otimes W$. That is, $\langle v_i\otimes w_j, v_k\otimes w_l\rangle_{V\otimes W}=\langle v_i,v_k\rangle_V\langle w_j\otimes w_l\rangle_W$, which is 0 unless $i=k$ and $j=l$.

So for any $\Phi=\sum_{i,j}\alpha_{i,j}(v_i\otimes w_j)$,

$$\langle\Phi,\Phi\rangle_{V\otimes W}=\left\langle\sum_{i,j}\alpha_{i,j}(v_i\otimes w_j),\sum_{k,l}\alpha_{k,l}(v_i\otimes w_j)\right\rangle_{V\otimes W}$$

$$=\sum_{i,j,k,l}\alpha_{i,j}\bar{\alpha}_{k,l}\langle v_i\otimes w_j,v_k\otimes w_l\rangle_{V\otimes W}$$

$$=\sum_{i,j}\alpha_{i,j}\bar{\alpha}_{i,j}=\sum_{i,j}|\alpha_{i,j}|^2$$

which can never be negative and is zero only if all $\alpha_{i,j}=0$.

C. What is the adjoint of $A\otimes B$?

By definition, the adjoint of $A\otimes B$ is the operator $(A\otimes B)^*$ that satisfies
$\langle (A\otimes B)(v\otimes w), v'\otimes w'\rangle_{V\otimes W}=\langle v\otimes w,(A\otimes B)^*(v'\otimes w')\rangle_{V\otimes W}$.
So we calculate:
$\langle (A\otimes B)(v\otimes w), v'\otimes w'\rangle_{V\otimes W}=\langle Av\otimes Bw, v'\otimes w'\rangle_{V\otimes W}=\langle Av,v'\rangle_V\langle Bw,w'\rangle_W$ (how $A\otimes B$ acts in $V\otimes W$, then definition of $\langle\ ,\ \rangle_{V\otimes W}$)
Then
$\langle Av,v'\rangle_V\langle Bw,w'\rangle_W=\langle v,A^*v'\rangle_V\langle w,B^*w'\rangle_W=\langle v\otimes w,A^*v'\otimes B^*w'\rangle_{V\otimes W}=\langle v\otimes w,(A^*\otimes B^*)(v'\otimes w')\rangle_{V\otimes W}$
(definition of adjoint in V and in W, then definition of $\langle\ ,\ \rangle_{V\otimes W}$; how $A^*\otimes B^*$ acts in $V\otimes W$)

Comparing both ends: $(A\otimes B)^*=A^*\otimes B^*$.

D. Now that we know how to define adjoints: Given P a projection in V and Q a projection in W, is $P\otimes Q$ a projection in $V\otimes W$?

$P\otimes Q$ is self-adjoint: $(P\otimes Q)^*=P^*\otimes Q^*=P\otimes Q$, since P and Q are self-adjoint.
$P\otimes Q$ is idempotent: $(P\otimes Q)(P\otimes Q)=P^2\otimes Q^2=P\otimes Q$, since P and Q are idempotent.

Q5: The dihedral group D_n and some of its representations.

The dihedral group D_n consists of the rotations and reflections of a regular n-gon. This group is generated by a rotation R of $\frac{2\pi}{n}$ and by a mirror M. The other mirror reflections are R^aM ($a=1,...,n-1$), and the identity. The group properties can all be derived from the relationships $R^n = M^2 = I$ (i.e., R is of order n and M is of order 2), and $MR = R^{n-1}M$ (a rotation followed by a mirror is the same as a mirror followed by a rotation in the opposite direction), without regard to a geometrical interpretation for R and M. It is a bit fussy -- even and odd values of n behave differently -- , but it is also a chance to work with groups via thee abstract relationships between their generators (here, R and M) – and to appreciate how useful it is to have a geometric interpretation.

A. *Determine whether all mirror reflections are in the same conjugate class as M. Since the group elements are I, R^a ($a=1,...,n-1$), and R^aM ($a=1,...,n-1$), it suffices to determine gMg^{-1} for each of these (other than the identity).*

gMg^{-1} for $g=R^a$: $gMg^{-1} = R^aM\left(R^a\right)^{-1} = R^aMR^{n-a}$. Applying $MR = R^{n-1}M$ to MR^{n-a} (i.e., applying it $n-a$ times, each time moving one copy of R to the left across M) yields
$MR^{n-a} = R^{n-1}MR^{n-a-1} = R^{2(n-1)}MR^{n-a-2} = ... = R^{(n-1)(n-a)}M = R^{-(n-1)a}M = R^aM$ (*).
So $R^aM\left(R^a\right)^{-1} = R^a\left(MR^{n-a}\right) = R^a\left(R^aM\right) = R^{2a}M$.
gMg^{-1} for $g=R^aM$: Same as for $g=R^a$, since $R^aMM\left(R^aM\right)^{-1} = R^a(R^aM)^{-1} = R^aMR^{-a}$.

So, conjugation of M by any group element can yield $R^{2a}M$, for any integer a. If n is odd, this yields all of the mirrors R^bM, as, for any integer b, $n-2a = b \pmod n$ always has an integer solution a. These are the mirrors that pass through any vertex and the midpoint of the opposite side.

But for n even, $n-2a = b \pmod n$ only has an integer solution a when b is even. That is, M is conjugate to R^2M, R^4M, … The other mirrors RM, R^3M, … are conjugate to each other but not to the first set of mirrors. These two sets correspond to the mirrors through a pair of opposite vertices, and the mirrors through a pair of midpoints of opposite sides.

B. *Determine the conjugate classes of the rotations. As in A, Since the group elements are I, R^a ($a=1,...,n-1$), and R^aM ($a=1,...,n-1$), it suffices to determine gR^kg^{-1} for each of these.*

For $g=R^a$, $gR^kg^{-1} = R^a$ since R commutes with powers of itself.
For $g=R^aM$, $gR^kg^{-1} = \left(R^aM\right)R^k\left(R^aM\right)^{-1}$. From part A (*), we had $MR^{n-a} = R^aM$, so

$$gR^kg^{-1} = \left(R^aM\right)R^k\left(R^aM\right)^{-1} = \left(MR^{n-a}\right)R^k\left(MR^{n-a}\right)^{-1} = MR^{n-a+k}\left(R^aM\right) = MR^kM$$

.Using part A (*) again, but as $MR^{n-k} = R^kM$,
yields $gR^kg^{-1} = MR^kM = M\left(MR^{n-k}\right) = R^{n-k} = R^{-k}$.

So each rotation R^k is conjugate to R^{-k}, i.e., they are conjugate in pairs except that for n even, $R^{n/2}$ is only conjugate to itself.

C. *Write out the conjugate classes for D_n.*

Collecting the results from A and B:
For n odd:

- the identity (one element)
- one class containing the mirrors (n elements)
- $\frac{n}{2}$ classes each containing two rotations $\{R^k, R^{-k}\}$, $k \in \{1,2,...,(n-1)/2\}$

For n even

- the identity (one element)
- one class of the "even" mirrors ($\frac{n}{2}$ elements, $\{M, R^2M, ..., R^{n-2}M\}$)
- one class of the "odd" mirrors containing ($\frac{n}{2}$ elements, $\{RM, R^3M, ..., R^{n-1}M\}$)
- $\frac{n}{2}-1$ classes each containing two rotations $\{R^k, R^{-k}\}$, $k \in \{1,2,...,n/2-1\}$
- One class containing the rotation by π, $\{R^{n/2}\}$.

D. *For definiteness, say that the n-gon has **one vertex pointing up**, and M is a reflection across the vertical axis. Consider elements of D_n as motions in the plane, and the corresponding 2-dimensional representation, say L, What is $\chi_L(R)$? What is $\chi_L(M)$? Can you construct other representations in a similar way?*

For R, this is a rotation by $\frac{2\pi}{n}$, so the corresponding matrix is $\begin{pmatrix} \cos\frac{2\pi}{n} & \sin\frac{2\pi}{n} \\ -\sin\frac{2\pi}{n} & \cos\frac{2\pi}{n} \end{pmatrix}$ and $\chi_L(R) = 2\cos\frac{2\pi}{n}$, its trace.

For M, the matrix is $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\chi_L(M) = 0$, its trace.

To construct other similar representations, we only require that the generator relationships $R^n = M^2 = I$ and $MR = R^{n-1}M$ hold for the matrices. We could have equally taken $\begin{pmatrix} \cos\frac{2\pi b}{n} & \sin\frac{2\pi b}{n} \\ -\sin\frac{2\pi b}{n} & \cos\frac{2\pi b}{n} \end{pmatrix}$ for the matrix corresponding to R, yielding distinct (but similar) representations for values of $b = 2, ..., \lfloor (n-1)/2 \rfloor$.

E. *Consider elements of D_n as permutations on the n edges and the corresponding n-dimensional representation, say E. What is $\chi_E(R)$? What is $\chi_E(M)$?*

For R: R is a cyclic permutation of the n edges, moving each edge to a different edge. As a permutation matrix, there are no 1's on the diagonal. So $\chi_E(R) = 0$.
For M: If n is odd, one edge always crosses the mirror. $\chi_E(M) = 1$. If n is even, no edges cross the mirror, so $\chi_E(M) = 0$

F. *As in E, but consider D_n as permutations on the n vertices.*

For n odd, the outcome is the same. But for n even, M leaves two vertices unchanged, so $\chi_V(M)=2$

G. There is a one-dimensional representation U that maps each $g \in D_n$ to the parity of the permutation on the edges corresponding to g. What is $\chi_U(R)$? What is $\chi_U(M)$?

n odd: R is a cyclic permutation of the n edges, so, if n is odd, this is an even permutation, and $\chi_U(R)=1$. *M swaps $(n-1)/2$ pairs of edges, leaving the edge opposite the top vertex unchanged, so, if $(n-1)/2$ is odd (i.e., $n=3,7,11,...$), then $\chi_U(M)=-1$ and otherwise (i.e., $n=5,9,13,...$) $\chi_U(M)=1$.*

n even: R is a cyclic permutation of the n edges, so, if n is even, this is an odd permutation, and $\chi_U(R)=-1$. *M swaps $\frac{n}{2}$ pairs of edges, so, if $n/2$ is odd (i.e., $n=6,10,14,...$), then $\chi_U(M)=-1$ and otherwise (i.e., $n=4,8,12,...$), $\chi_U(M)=1$.*