Linear Transformations and Group Representations

Homework #2 (2024-2025), Questions

Q1: Orthogonality of the characters of SO(3)

Setup: As detailed in class notes, the group of rotations of a sphere in 3-space, $SO(3)$, has irreducible representations L_m, of dimension $2m+1$, for each $m \in \{0,1,2,...\}$. In L_m. a rotation by θ about the "z" axis is given by

$$\begin{pmatrix} 1 & & & & & & \\ & \cos\theta & \sin\theta & & & & \\ & -\sin\theta & \cos\theta & & & & \\ & & & \ddots & & & \\ & & & & \cos m\theta & \sin m\theta \\ & & & & -\sin m\theta & \cos m\theta \end{pmatrix},$$

which, after a coordinate change, is

$$\begin{pmatrix} 1 & & & & & \\ & e^{i\theta} & & & & \\ & & e^{-i\theta} & & & \\ & & & \ddots & & \\ & & & & e^{im\theta} & \\ & & & & & e^{-im\theta} \end{pmatrix}.$$

So the character for the conjugate class of rotations by an angle $\theta \in [0,\pi]$ is given by $\chi_{L_m}(R_\theta) = \sum_{k=-m}^{m} e^{ik\theta}$.

We further stated that these characters are orthonormal, if they are properly weighted by the "mass " of their conjugate classes: $\int_0^\pi \chi_{L_m}(R_\theta)\overline{\chi_{L_n}(R_\theta)}w(\theta)d\theta = \begin{cases} 0, & m \neq n \\ 1, & m = n \end{cases}$, where $w(\theta)$ is the weighting of the conjugate class θ, $w(\theta) = \frac{1}{\pi}(1-\cos\theta)$. Here we demonstrate this orthonormality. Several steps but each should be straightforward.

A. For any real-valued function $f(\theta)$ that can be written as $f(\theta) = \sum_{k=-m}^{m} f_k e^{ik\theta}$, determine $\int_0^\pi f(\theta)d\theta$ in terms of the f_k.

B. Given any two functions that are of the form specified in part A (say, $f(\theta) = \sum_{k=-m}^{m} f_k e^{ik\theta}$ and $g(\theta) = \sum_{l=-n}^{n} g_l e^{il\theta}$), express the results of multiplying them as a third such function.

C. Write $w(\theta) = \frac{1}{\pi}(1-\cos\theta)$ in the form specified in part A.

D. For any non-constant real-valued function $f(\theta)$ of the form specified in part A, compute $\int_0^{\pi} f(\theta)w(\theta)d\theta$ in terms of the f_k.

E. For $f(\theta) = \chi_{L_m}(R_\theta)$, $g(\theta) = \chi_{L_n}(R_\theta)$, and $z(\theta) = f(\theta)g(\theta)$, determine z_k so that $f(\theta)g(\theta) = \sum_{k=-(m+n)}^{m+n} z_k e^{ik\theta}$ and, with the results of part D, demonstrate orthonormality.