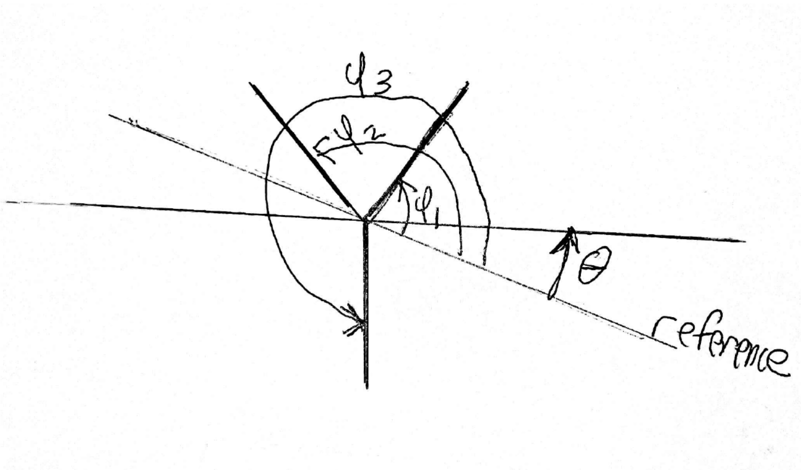
Multivariate Analysis

Homework #2 (2024-2025), Answers

These problems examine the performance of ICA variants in several illustrative "edge cases."

Q1. Moments of a toy distribution

Consider a bivariate distribution $p(\vec{x})$ that is concentrated on S "spokes" emanating from the origin, each of unit length, and is uniformly distributed along those spokes. The directions of the spokes are specified by unit vectors $\vec{v}_k$, $k \in \{1,...,S\}$, each at an angle φ_k with respect to some reference direction. Project this distribution onto a line at an angle θ with respect to the same reference. See diagram below for $S=3$; $p(\vec{x})$ is concentrated on the solid "Y".



A) Write the n th moment $M_n(\theta)$ of the resulting distribution in terms of the φ_k. B) Under what conditions on the $\vec{v}_k$ are the means of all those distributions is zero, i.e., that $M_1(\theta)=0$ for all θ? C) What can one say about the shape of $M_2(\theta)$, i.e., about the directions θ for which $M_2(\theta)$ is maximized or minimized?

A) The distribution is a sum of S components, each with a mass of $\frac{1}{S}$. Each component is uniformly distributed on the projection of one of the spokes onto the reference – which is a segment of length $\cos(\varphi_k - \theta)$, with the convention that negative lengths are in the opposite direction to positive lengths.

So $M_n(\theta) = \frac{1}{S}\sum_{k=1}^{S}\int_0^1 \left(r\cos(\varphi_k-\theta)\right)^n dr = \frac{1}{S}\sum_{k=1}^{S}\cos^n(\varphi_k-\theta)\int_0^1 r^n dr = \frac{1}{S(n+1)}\sum_{k=1}^{S}\cos^n(\varphi_k-\theta)$.

B) For M_1, this is $M_1(\theta) = \frac{1}{2S}\sum_{k=1}^{S}\cos(\varphi_k-\theta)$. With $\vec{z}(\theta)$ a unit vector in the direction θ, this is $M_1(\theta) = \frac{1}{2S}\sum_{k=1}^{S}\vec{z}(\theta)\bullet\vec{v}_k = \frac{1}{2S}\left(\vec{z}(\theta)\bullet\sum_{k=1}^{S}\vec{v}_k\right)$. As θ ranges over $[0,2\pi)$, the unit vectors $\vec{z}(\theta)$ form an overcomplete basis (or it suffices to consider any two θ 's that don't differ by π). So $\vec{z}(\theta)\bullet\sum_{k=1}^{S}\vec{v}_k$ can only be zero for all θ if $\sum_{k=1}^{S}\vec{v}_k = 0$.

C) For M_2, this is $M_2(\theta)=\frac{1}{3S}\sum_{k=1}^{S}\cos^2(\varphi_k-\theta)=\frac{1}{3S}\sum_{k=1}^{S}\frac{1+\cos\left(2(\varphi_k-\theta)\right)}{2}$. As a function of θ, this is a constant plus a sum of sinusoids of period π. So, as θ ranges over $[0,2\pi)$, it will either be constant, or have two maxima spaced by π, and two minima halfway in between. That is, the directions of maximum and minimum variance will always be orthogonal.

Note that this holds for any distribution of spokes on the unit disc. A similar argument shows that it holds for any distribution of spokes on a disc of any size, and that it holds for a superposition of such distributions. So, it holds for any distribution – so we've shown that the directions of maximum and minimum variance are always orthogonal.

Q2. Consider a special case of the distribution in Q1, of four equally-spaced, equally-weighted spokes, with one spoke at an angle of 0 to the reference line. Compute $M_2(\theta)$, $M_3(\theta)$ and $M_4(\theta)$, and determine their maxima and minima.

With $\varphi_k=\frac{k-1}{4}(2\pi)$, $M_n(\theta)=\frac{1}{4(n+1)}\sum_{k=1}^{4}\cos^n(\varphi_k-\theta)=\frac{1}{4(n+1)}\left((\cos\theta)^n+(\sin\theta)^n+(-\cos\theta)^n+(-\sin\theta)^n\right)$.

So for $n=2$:

$M_2(\theta)=\frac{1}{4\bullet3}\left(2(\cos\theta)^2+2(\sin\theta)^2\right)=\frac{1}{6}\left(\cos^2\theta+\sin^2\theta\right)=\frac{1}{6}$, a constant.

For n odd, this is zero.

For $n=4$:

$M_4(\theta)=\frac{1}{4\bullet5}\left(2(\cos\theta)^4+2(\sin\theta)^4\right)=\frac{1}{10}\left(\cos^4\theta+\sin^4\theta\right)$

$=\frac{1}{10}\left(\left(\cos^2\theta+\sin^2\theta\right)^2-2\cos^2\theta\sin^2\theta\right)$,so there are maxima at 0, $\pi/2$, π, and $3\pi/2$ and

$=\frac{1}{10}\left(1-\frac{1}{2}\sin^2 2\theta\right)=\frac{1}{10}\left(1-\frac{1}{4}(1-\cos4\theta)\right)=\frac{3}{40}+\frac{1}{40}\cos4\theta$

minima halfway in between.

Q3. Consider a special case of the distribution in Q1, of three equally-spaced, equally-weighted spokes, with one spoke at an angle of 0 to the reference line. Compute $M_2(\theta)$, $M_3(\theta)$ and $M_4(\theta)$, and determine their maxima and minima.

With $\varphi_k=\frac{k-1}{3}(2\pi)$,

$$M_n(\theta)=\frac{1}{3(n+1)}\sum_{k=1}^{3}\cos^n(\varphi_k-\theta)=\frac{1}{4(n+1)}\left((\cos\theta)^n+\left(\cos(\frac{2\pi}{3}-\theta)\right)^n+\left(\cos(\frac{4\pi}{3}-\theta)\right)^n\right)$$

$$=\frac{1}{3(n+1)}\left((\cos\theta)^n+\left(-\frac{1}{2}\cos\theta+\frac{\sqrt{3}}{2}\sin\theta\right)^n+\left(-\frac{1}{2}\cos\theta-\frac{\sqrt{3}}{2}\sin\theta\right)^n\right).$$

For $n=2$:

$$M_2(\theta)=\frac{1}{9}\left((\cos\theta)^2+\left(-\frac{1}{2}\cos\theta+\frac{\sqrt{3}}{2}\sin\theta\right)^2+\left(-\frac{1}{2}\cos\theta-\frac{\sqrt{3}}{2}\sin\theta\right)^2\right),$$

$$=\frac{1}{9}\left((\cos\theta)^2+2\bullet\frac{1}{4}\cos^2\theta+2\bullet\frac{3}{4}\sin^2\theta\right)=\frac{1}{9}\bullet\frac{3}{2}\left(\cos^2\theta+\sin^2\theta\right)=\frac{1}{6}$$

a constant. (This is also the average of $M_2(\theta)$ in Q2).

For $n=3$:

$$M_3(\theta)=\frac{1}{12}\left((\cos\theta)^3+\left(-\frac{1}{2}\cos\theta+\frac{\sqrt{3}}{2}\sin\theta\right)^3+\left(-\frac{1}{2}\cos\theta-\frac{\sqrt{3}}{2}\sin\theta\right)^3\right)$$

$$=\frac{1}{12}\left((\cos\theta)^3-2\bullet\frac{1}{8}\cos^3\theta-2\bullet 3\bullet\frac{3}{8}\cos\theta\sin^2\theta\right)=\frac{1}{12}\frac{3}{4}\left(\cos^3\theta-3\cos\theta\sin^2\theta\right).$$

$$=\frac{1}{8}\left(\cos\theta\left(\cos^2\theta-\sin^2\theta\right)-\sin\theta\left(2\sin\theta\cos\theta\right)\right)=\frac{1}{8}\left(\cos\theta\cos 2\theta-\sin\theta\sin 2\theta\right)=\frac{1}{8}\cos 3\theta$$

(The answer had to have threefold symmetry.) So there are maxima at $\theta=\frac{k-1}{3}(2\pi)$ and minima halfway in between.

For $n=4$:

$$M_4(\theta)=\frac{1}{15}\left((\cos\theta)^4+\left(-\frac{1}{2}\cos\theta+\frac{\sqrt{3}}{2}\sin\theta\right)^4+\left(-\frac{1}{2}\cos\theta-\frac{\sqrt{3}}{2}\sin\theta\right)^4\right)$$

$$=\frac{1}{15}\left((\cos\theta)^4+2\bullet\frac{1}{16}\cos^4\theta+2\bullet 6\bullet\frac{3}{16}\cos^2\theta\sin^2\theta+2\bullet\frac{9}{16}\sin^4\theta\right),$$

$$=\frac{1}{15}\left(\frac{9}{8}\cos^4\theta+\frac{9}{4}\cos^2\theta\sin^2\theta+\frac{9}{8}\sin^4\theta\right)=\frac{1}{15}\bullet\frac{9}{8}\left(\cos^2\theta+\sin^2\theta\right)^2=\frac{3}{40}$$

a constant. (This is also the average of $M_4(\theta)$ in Q2).

Q4. Consider a special case of the distribution in Q1, of five equally-spaced, equally-weighted spokes. What can you say about the behavior of $M_2(\theta)$, $M_3(\theta)$ and $M_4(\theta)$?

The dependence of any $M_n(\theta)$ must have fivefold rotational symmetry, since the distribution has this symmetry. But it also must be a sum of squares of trigonometric functions (for $M_2(\theta)$), of cubes of trigonometric functions (for $M_3(\theta)$), or of fourth powers (for $M_4(\theta)$). So all of these must be independent of θ.