Exam, 2024-2025 Questions

Overview

Q1: 4 parts
Q2: 4 parts
Q3: 4 parts
Q4: 3 parts
Q5: 8 parts
Q6: 4 parts

Total: 27 parts (with widely varying levels of difficulty). Do any 14.

1. Group theory: Normal subgroups, representations

Given a finite group G with a normal subgroup H that is not all of G:

A. Consider a coset Ha. Show that every element of the form hah' ($h' \in H$) is also an element of the coset Ha.

B. Now consider the set C of distinct cosets. Right-multiplication by group elements is a permutation on C, and therefore, yields a representation L_C of G as permutation matrices; the dimension of this representation is $|C| = \frac{|G|}{|H|}$. What is its character?

C. Is L_C irreducible? Why or why not?

D. If a finite group G has any subgroup H whose size is exactly half of that of G, find a nontrivial irreducible representation.

2. Group theory: Conjugate classes, automorphisms, representations

The "quaternion group" Q is an 8-element group that can be defined as follows. Its elements are $\{1,-1,i,-i,j,-j,k,-k\}$. 1 and -1 behave in the standard fashion, i.e., multiplication by 1 is the identity, and multiplication by -1 is a sign change. For the other elements: all of their squares are -1, and they multiply as follows: $ij = k$, $jk = i$, $ki = j$, $ji = -k$, $kj = -i$, $ik = -j$.

A. What are the conjugate classes of Q?

B. What are the inner automorphisms of Q?

C. What all the automorphisms of Q?

D. What is the character table of Q? Use part B, or, part D of Question 1, to find the one-dimensional representations.

3. Fourier analysis as a unitary transformation; generating functions

In its standard form, Fourier transformation is almost a unitary transformation – the inner product of two functions differs from the inner product of their Fourier transforms by a factor of 2π (Parseval's Theorem). We can make it unitary by a slightly nonstandard formulation, which presents a nearly symmetric relationship between complex-valued functions on the line and their Fourier transforms:

$$\widehat{f}(x) = (Sf)(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ixu} f(u)du \tag{1}$$

and

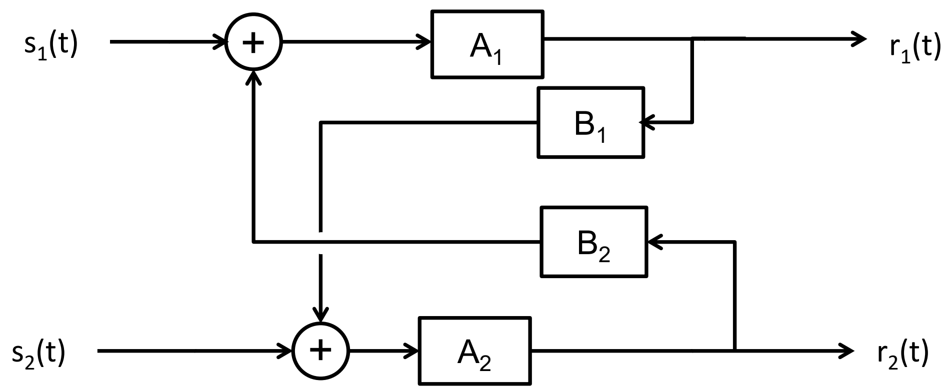
$$f(x) = \left(S^{-1}\widehat{f}\right)(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ixu} \widehat{f}(u)du \,. \tag{2}$$

In this formulation, Fourier transformation is truly unitary: $\int_{-\infty}^{\infty} \widehat{f}(x)\overline{\widehat{g}(x)}dx = \int_{-\infty}^{\infty} f(x)\overline{g(x)}dx$ .We write this "unitarized" Fourier transformation as an operator (i.e., $\widehat{f}(x) = (Sf)(x)$), to emphasize this viewpoint .Here, we find the eigenvalues and eigenvectors of S, and use this to define operations that are "fractional" Fourier transforms.

A. Hermite polynomials $h_n(x)$ may be defined via a generating function $\sum_{n=0}^{\infty} \frac{h_n(x)}{n!} t^n \triangleq h(x,t) = \exp\left(xt - \frac{t^2}{2}\right)$. Instead we consider the Hermite functions, defined (here) by $H_n(x) = \frac{1}{2^{n/2}} h_n(x\sqrt{2}) \exp\left(-\frac{x^2}{2}\right)$. Determine the generating function $H(x,t) \triangleq \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} t^n$.
B. What is $SH(x,t)$, the unitarized Fourier transform (eq. 1), of the generating function $H(x,t)$ (with respect to x)?
C. By comparing the generating functions $H(x,t)$ and $SH(x,t)$, determine eigenvalues and eigenvectors of S.
D. Provide a recipe for an operator R that is the operator-k th-root of a Fourier transform, namely, an operator R for which $R^k = S$ (other than the trivial $R = S$). (Technical: the recipe should make sense for all functions f that can be written as an absolutely convergent sum of Hermite functions.)

4. Input-output systems, noise, covariance

Consider the two-input, two-output system below, where the transformations A_i and B_i are linear.



A. Determine the transfer functions $\hat{L}_{ij}$ between input s_i and output r_j

B. If s_1 and s_2 are uncorrelated Gaussian white noises with unit spectral density, what is the power spectrum of r_1 and of r_2?

C. What is the coherence of r_1 and r_2?

5. Principal components analysis

Consider a data matrix X with elements $x_{i,j}$ (R rows, C columns, $C \gg R$), in which all rows and columns are linearly independent (so there are R principal components). Indicate to what extent these manipulations can change the number of principal components.

A. The rows of X are permuted.

B. The mean of each column is subtracted from each element of X, i.e., $x'_{i,j} = x_{i,j} - \frac{1}{R}\sum_k x_{k,j}$.

C. The mean of each row is subtracted from each element of X, i.e., $x'_{i,j} = x_{i,j} - \sum_k x_{i,k}$,

D. Each element of X is replaced by its square.

E. Each element of X is replaced by its first N Fourier components (not including DC).

F. New rows of X are adjoined to X, with each new row equal to the sum of the first N Fourier components (not including DC) of one row in the original X.

G. Each element of X is replaced by $\max(0, x_{i,j})$, and new rows are adjoined, given by $\min(0, x_{i,j})$. That is, each row is split into two rows, one containing only the positive values and one containing only the negative values.

H. New rows are adjoined to X, consisting of averages of random-with-replacement selections from X.

6. Maximum-entropy distributions

A. What is the functional form for a maximum-entropy distribution for a continuous non-negative scalar variable, with constrained (arithmetic) mean and geometric mean?

B. Show how the unique parameter values for the distribution can be determined from the given arithmetic and geometric means.

C. As in A, but now, constrained arithmetic mean and root-mean-square.

D. Determine the maximum-entropy distribution for a bivariate process $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, in which both variables are constrained to be non-negative, and the mean value of their sum is constrained to be s. Are the variables independent?