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Discriminable Textures with Identical Buffon Needle Statistics

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Abstract. A method is presented for generating pairs of textures for which the statistics of intersection with any collinear set of points placed at random on either texture are the same. The constraint that such "Buffon needle statistics" be identical is stronger than identity of second-order statistics. Nonetheless, many such texture pairs are effortlessly distinguishable. An example is given of such a texture pair, whose components are composed of either congruent ellipses or circles of various radii. The discriminability of such texture pairs implies that adequate models for human texture preception must contain local nonlinearities which receive input from non-collinear points.

1. Introduction

In 1962, Julesz conjectured that "textures cannot be effortlessly discriminated that agree in their second-order statistics". This conjecture is equivalent to the statement that, prior to the decision-making threshold, effortless texture discrimination is a fundamentally linear process. The vast majority of experimental evidence supports this conjecture (Julesz et al., 1973; Julesz, 1975). Thus, the few rare exceptions that have recently been found (Caelli and Julesz, 1978; Caelli et al., 1978; Julesz et al., 1978) are of interest because they may be psychophysical indications of nonlinear feature detectors that participate in texture perception.

This report desribes another pair of readily distinguishable textures which, in addition to sharing identical second-order statistics, also share certain statistics of higher order. Specifically, the statistics of intersection with the textures of any set of collinear points, dropped at random on either texture, are identical. As the statistics of the intersection of random linear figures with a fixed pattern were first considered by Buffon (1773, 1777) in his famous "needle problem", we refer to statistics of the sort described above as "Buffon needle statistics". Thus the textures described in this note are "iso-Buffon needle", not just "iso-dipole". That these textures are effortlessly discriminable places a lower bound on the complexity of an algorithm to mimic human texture perception (Marr, 1976).

2. Construction of a Discriminable Iso-Buffon Needle Texture Pair

We now present a simple example of a texture pair that is effortlessly discriminable, despite identity of the Buffon needle statistics (Fig. 1).

A B

Fig. 1. Two textures with identical Buffon needle statistics. Texture A is composed of congruent ellipses at random position and orientation. Texture B is composed of circles at random position whose radii are distributed according to the weighting (1)

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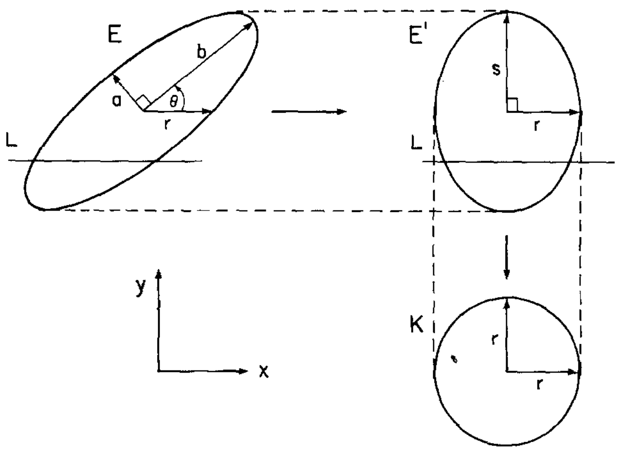

Fig. 2. Construction of circles whose Buffon needle statistics duplicate those of ellipse E at angle θ to intersecting line L. Ellipse E is transformed by an affine transformation into ellipse E', whose axes are parallel and perpendicular to L. Ellipse E has the same area as ellipse E. The circle K is obtained from ellipse E' by a rescaling along the vertical coordinate

Texture A is composed of ellipses with semiaxes a and b. The centers of the ellipses are drawn from a two-dimensional Poisson process, and the orientations of the ellipses are chosen at random from a uniform distribution. The union of the interiors of the ellipses comprises the dark regions of the texture. Texture B is composed of circles of various radii. The number density (the average number per unit area) of circles in texture B is equal to the number density of ellipses in texture A. The radii r lie in the range $a \leqq r \leqq b$, where a and b are the semiaxes of the ellipses in texture A. The fraction of circles whose radius lies between r and $r+dr$ is given by the weighting function

$$w(r)dr = \begin{cases} \dfrac{ab}{b-a} \cdot \dfrac{dr}{r^2}, & a \leqq r \leqq b \\ 0, & \text{otherwise}. \end{cases} \tag{1}$$

Thus, the frequency with which circles appear in the texture is inversely proportional to the square of their radii. The circles are positioned according to a two-dimensional Poisson process, and the union of their interiors constitutes the dark phase of Texture B.

An interesting consequence of (1) is that the total area of those circles whose radii lie within the interval $r_1 \leqq r \leqq r_2$ depends only on the difference $r_2 - r_1$.

3. Proof that theTwo Textures Have Identical Buffon Needle Statistics

In this section, we prove that the textures of Fig. 1 have identical Buffon needle statistics. That is, all the statistics of intersection of a line segment dropped at random on each texture are identical.

We begin by proving a basic theorem about the ellipses and circles that constitute the skeleton of textures such as those in Fig. 1.

Theorem. *Let L be a line segment at random orientation and position and let X be a set of points $\{x_1, \ldots, x_N\}$ on L and let $\mathscr{P}$ be a partition of X into M disjoint subsets $X_1, \ldots, X_M$. Let $\mathbf{P}_A(X, \mathscr{P}, \varepsilon)$ be the probability that the intersection of L with the ellipses of semiaxes $a<b$ at random position and orientation occur only at points $y_1, \ldots, y_N$ satisfying $|y_j - x_j| < \varepsilon$ $(1 \geqq j \geqq N)$, with the points y_j, y_k belonging to the same ellipse if and only if x_j and x_k are drawn from a single element X_m of the partition $\mathscr{P}$. Let $\mathbf{P}_B(X, \mathscr{P}, \varepsilon)$ be the probability that the intersection of L with the circles of random position and radii distributed according to (1) occur only at points $z_1, \ldots, z_N$ satisfy $|z_j - x_j| < \varepsilon$ $(1 \leqq j \leqq N)$, with the points z_j, z_k belonging to the same circle if and only if x_j and x_k are drawn from a single element X_m of the partition $\mathscr{P}$.*

Then $\mathbf{P}_A(X, \mathscr{P}, \varepsilon) = \mathbf{P}_B(X, \mathscr{P}, \varepsilon)$.

Proof. Since both texture skeletons are formed from Poisson-distributed elements it suffices to consider intersections with only a single ellipse or circle ($M=1$). Since both texture skeletons are isotropic, it also suffices to consider lines L of one fixed orientation.

We now show that the statistics of intersection of a horizontal line L and an ensemble of ellipses at an orientation θ are identical to those of the line L and an appropriate ensemble of circles all of the same radius, with number density equal to $w(r)$ times the number density ϱ of the ellipses. This radius r depends on the semiaxes a and b, and on the angle θ between the major axis and the horizintal.

Our demonstration rests on the transformation of ellipses into circles by a pair of linear deformations of the plane, whose effects on line-intersection statistics are easily analyzed. Figure 2 shows a typical configuration of a horizontal line L intersecting an ellipse E at orientation θ. The ellipse E' is derived from the ellipse E by an affine transformation

$$(x, y) \to (x + \alpha y, y)$$

where α is chosen so that the semiaxes r and s of the new ellipse E' are parallel and perpendicular to the line L. Since the affine transformation preserves the length of L and of all horizontal slices of the ellipse E, the resulting ellipse E' has the same statistics of intersection with horizontal lines. Also, since affine transformations preserve area, the products of the semiaxes of the ellipses E and E' are equal:

$$ab = rs. \tag{2}$$

Now we observe that the intersection statistics of the line L with E' are identical to the intersection statistics of this line with a circle K of radius r, except that intersections with the circle are less frequent because of the smaller vertical cross-section of the circle. However, one may compensate for this difference by choosing a density for the circles of radius r whose ratio $w(r)$ to the density of ellipses at orientation θ is exactly the ratio of vertical cross-sections, $\frac{s}{r}$. Ellipses at different orientations may be similarly replaced by equivalent circles of different radii, as determined by the construction shown in Fig. 2. Since the orientation distribution of the ellipses is uniform, we find

$$w(r)=c\frac{s}{r}.$$

Using (2),

$$w(r)=c\frac{ab}{r^2}.$$

The constant c determines the relative density of the circles compared to that of the ellipses. This constant may be evaluated by observing that equal intersection statistics imply equal total area enclosed by the texture skeletons, by Cavalieri's principle (see Underwood, 1970, pp. 27–29). Therefore,

$$\varrho\pi ab=\int_{r_{\min}}^{r_{\max}}\varrho w(r)\pi r^2 dr=\varrho\pi abc\int_{r_{\min}}^{r_{\max}}dr,$$

or

$$1=c\int_{r_{\min}}^{r_{\max}}dr.$$

Inspection of Fig. 2 shows that $r_{\min}$, the minimum value of r, is equal to the short semiaxis a, and occurs when $\theta=90°$. Also, $r_{\max}$, the maximum value of r, is equal to the long semiaxis b, and occurs when $\theta=0°$. Therefore,

$$1=c\int_a^b dr, \quad \text{which implies} \quad c=\frac{1}{b-a}, \quad \text{and}$$

$$w(r)=\frac{ab}{b-a}\cdot\frac{1}{r^2}, \quad \text{as asserted in (1).}$$

Thus, we have shown that the Buffon needle statistics of the skeletons of textures A and B are identical. Q.E.D.

We have

$$\int_a^b w(r)dr=1;$$

this implies that the number density of circles equals the number density of ellipses.

The dark regions of the textures of Fig. 1 consist of all points which lie in the interiors of any ellipse or circle in their skeletons. The Buffon needle statistics are equal by virtue of the following:

Corollary. *Let $T_A(n)$ be the union of all points contained within exactly n ellipses of semiaxes $a<b$ at random positions and orientations. For any set of integers S, define $T_A(S)=\bigcup_{n\in S}T_A(n)$. Let $T_B(n)$ be the union of all points contained within exactly n circles of random position, and radii distributed according to* (1), *and let $T_B(S)=\bigcup_{n\in S}T_B(n)$. Then, the intersection statistics of a line segment L with $T_A(S)$ are identical to those of L with $T_B(S)$.*

Proof. An intersection of L with $T_A(S)$ or $T_B(S)$ is a sequence of segments contained in L. The statistics of these sub-segments are determined by the statistics of their endpoints. By the Theorem above, these are identical. Q.E.D.

Taking $S=\{1,2,3,\ldots\}$ in the Corollary implies that the textures A and B of Fig. 1 have identical Buffon needle statistics.

4. Generalizations of the Construction of Iso-Buffon Needle Textures

There are several immediate generalizations of the construction of Sects. 2 and 3. We illustrate these more complex texture pairs by example.

1. Iso-Buffon needle texture pairs may be constructed by using any set S in the Corollary. For example, one may darken the points that are in the interior of exactly an odd number of ellipses or circles, which amounts to choosing $S=\{1,3,5,\ldots\}$.

2. Iso-Buffon needle texture pairs may be constructed by replacing uniformly dark regions of the textures $T_A(S)$ and $T_B(S)$ by a wide variety of more complex textures.

3. The simple form of (1) permits the construction of iso-Buffon needle textures intermediate between the ellipse texture A and the circle texture B. Choose a number h between a and b. We construct a third texture of two kinds of ellipses, E_{ah} of semiaxes a and h, and E_{hb} of semiaxes h and b. Let $\frac{h-a}{b-a}\frac{b}{h}$ of the ellipses be of the type E_{ah}, and the remaining $\frac{b-h}{b-a}\frac{a}{h}$ of them be of type E_{hb}. The equivalent distribution of circle sizes for this texture is

$$w'(r)=\frac{h-a}{b-a}\frac{b}{h}w_{ah}(r)+\frac{b-h}{b-a}\frac{a}{h}w_{hb}(r),$$

where

$$w_{pq}(r) \equiv \begin{cases} \frac{pq}{q-p} \cdot \frac{1}{r^2}, & p \leqq r \leqq q \\ 0, & \text{otherwise}. \end{cases}$$

Here p is either a or h, and q is either h or b. But except at $r=h$, $w'(r)$ is identically equal to $w_{ab}(r)$, which is the same as $w(r)$ of (1). Thus, the two-ellipse texture has identical line intersection statistics with the circle texture of (1), and its corresponding single-ellipse texture[1].

This procedure may be repeated to generate a pair of equivalent texture skeletons of any two partitions of the interval from a to b. These texture skeletons may then be filled in according to procedures 1) or 2) to generate a large family of iso-Buffon needle texture pairs.

5. Discussion

We have presented an algorithm for the generation of a large family of iso-Buffon needle texture pairs. These texture pairs, whose detailed intersection statistics with line segments are identical, are a fortiori iso-dipole textures. That the textures comprising at least one of these pairs (Fig. 1) are effortlessly discriminable is therefore a counterexample to the conjecture of Julesz (1962; Julesz et al., 1973) that effortless discrimination requires distinct second-order statistics.

Three exceptions to this conjecture have already been found by the four-disk method and its extensions (Caelli and Julesz, 1978; Caelli et al., 1978). These exceptions have been interpreted as evidence for three classes of nonlinear feature detectors in the human visual system: quasi-collinearity (B_1), corner (B_2) and closure (B_3). A fourth exception (Julesz et al., 1978) and the exception presented above are somewhat different in nature from the original B_1, B_2, and B_3 series: both of these exceptions obey *stricter* statistical constraints than the original second-order one, and both are more difficult to interpret in terms of local features. For example, in the iso-trigon texture counterexample (Julesz et al., 1978), discrimination may be based on apparent granularity, or apparent edge orientation, or closure vs. non-closure. In the present example, in which statistics are identical to all orders provided that the sampling points are collinear, it is not immediately obvious whether discrimination is based on "orientation granularity", or border length, or curvature dispersion or some other qualitative difference between the two textures. The generalizations discussed in Sect. 4 of the texture pairs of Fig. 1 may be useful in deciding which of these interpretations is most appropriate.

The exceptions to the iso-dipole conjecture are also interesting because they provide lower bounds on the complexity of any algorithmic model of human texture perception (for example that of Marr, 1976). Thus, the presence of even a single exception requires that an adequate model have a local nonlinearity. The iso-trigon exception shows that the local nonlinearity must be formally of high order. The present iso-Buffon needle exception demonstrates that there must be a local nonlinearity that receives input from non-collinear points. Thus, the paradigm of texture discrimination serves as a useful tool in elucitading qualitative features of the human visual system.

[1] The singularity at $r=h$, where $w'(h)=2w_{ab}(h)$, is of no consequence because the functions w' and w_{ab} differ only on a set of measure zero

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