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A Relation Between the Akaike Criterion and Reliability of Parameter Estimates, with Application to Nonlinear Autoregressive Modelling of Ictal EEG

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The Akaike minimum information criterion provides a means to determine the appropriate number of lags in a linear autoregressive model of a time series. We show that the Akaike criterion is closely related to the reliability estimates of successively determined parameters of a linear autoregressive (LAR) model. A similar criterion may be applied to determine whether the addition of a nonlinear term to an LAR model provides a statistically significant improvement in the description of the time series. As an example, we use this method to identify quadratic contributions to a nonlinear autoregressive characterization of a typical 3/s spike and wave seizure discharge.

Keywords—*Autoregressive models, EEG, Information theory, Nonlinear analysis.*

INTRODUCTION

Autoregressive (AR) modelling is a powerful technique for the efficient representation of time-series data (4,27). To construct an AR description of a time series, the investigator must decide on the number of terms to be included. Since coefficients of the included terms are determined by a least-squares procedure, the unexplained variance necessarily decreases as the number of terms is increased. Thus, a minimum-variance criterion cannot be used to determine the number of terms, which should be included in the AR model. Rather, in order that the addition of a term be justified, the reduction in unexplained variance must be sufficiently large. Akaike (1) advanced an information-theoretic criterion for deciding just how large a variance reduction is

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necessary. His argument was based on a maximum-likelihood estimate of model parameters, which necessarily postulated that the driving terms (innovations) of the model are Gaussian-distributed. Thus, in a strict sense, the Akaike criterion is limited to linear AR (LAR) models.

In this paper, we deduce reliability estimates for both linear and nonlinear terms in an AR model. We demonstrate that for a fully linear AR model, a criterion based on reliability estimates essentially coincides with that of Akaike. This provides a rationale for using the reliability estimate criterion for nonlinear AR (NLAR) models, where the Akaike criterion cannot rigorously be applied. As an example, we apply this technique to a segment of electroencephalogram (EEG) recorded during a seizure discharge, and demonstrate how the NLAR formalism suggests features of the underlying EEG dynamics.

THEORY

The NLAR Model

We formulate the NLAR model for a time series composed of samples y_n as follows:

$$y_n = \sum_{j=1}^{J} c_j f_j(y_{n-1}, y_{n-2}, \ldots, y_{n-L}) + x_n \ . \tag{1}$$

The quantities x_n are assumed to be drawn independently from a probability distribution $p(x)$, which we initially assume is a Gaussian distribution of variance U.

It is convenient to represent the current and previous values of the time series at step n by $Y_n = (y_n, y_{n-1}, \ldots)$, and to use the notation $f_j(Y_n) = f_j(y_{n-1}, y_{n-2}, \ldots, y_{n-L})$. In this analysis we will consider the time series $\{y_n\}$ to form a cyclic sequence of length N. Quantities such as $f_1, \ldots, f_J$ and $c_1, \ldots, c_J$ will be grouped as column-vectors $\mathbf{f}$ and $\mathbf{c}$. With these conventions, Eq. 1 can be rewritten

$$y_n = \mathbf{f}(Y_n)^T\mathbf{c} + x_n \ . \tag{2}$$

Our goal is to determine how estimates of $\mathbf{c}_{\text{est}}$ from time series generated by Eq. 2 will differ from their exact values $\mathbf{c}$. We assume that $\mathbf{f}$ is known, and we focus on limiting behavior as N becomes large in comparison with the number J of terms in Eq. 2 and the maximum lag L relevant to any of the f_j. We will use $\langle\ \rangle_{\text{est}}$ to indicate averages over an empirical data set and $\langle\ \rangle$ to indicate exact expected values (averages over the larger ensemble from which the empirical data set is drawn).

Reliability of Estimated Parameters

Estimates of $\mathbf{c}$ may be derived from an empiric time series $\{y_n\}$ by minimizing the unexplained variance $R = (1/N)\Sigma x_n^2$. Under the hypothesis that the x_n are distributed in a Gaussian fashion with mean zero, this estimate is the maximum-likelihood estimate for $\mathbf{c}$. Yule–Walker equations (7) for $\mathbf{c}_{\text{est}}$ may, therefore, be obtained by setting $\partial R/\partial c_j = 0$ for each c_j:

$$\langle \mathbf{f}(Y_n)y_n \rangle_{\text{est}} = \langle \mathbf{f}(Y_n)\mathbf{f}(Y_n)^T \rangle_{\text{est}}\mathbf{c}_{\text{est}} \ . \tag{3}$$

We denote the column vector $\langle \mathbf{f}(Y_n)y_n\rangle_{\text{est}}$ by $\mathbf{b}_{\text{est}}$ and the symmetric matrix $\langle \mathbf{f}(Y_n)\mathbf{f}(Y_n)^T\rangle_{\text{est}}$ by $\mathbf{F}_{\text{est}}$. Provided $\mathbf{F}_{\text{est}}$ is nonsingular, estimates $\mathbf{c}_{\text{est}}$ may be calculated by

$$\mathbf{c}_{\text{est}} = \mathbf{F}_{\text{est}}^{-1}\mathbf{b}_{\text{est}} \ . \tag{4}$$

A similar argument shows that there is an analogous relationship between the exact values $\mathbf{c}$ and the exact values $\mathbf{b} = \langle \mathbf{f}(Y_n)y_n\rangle$ and $\mathbf{F} = \langle \mathbf{f}(Y_n)\mathbf{f}(Y_n)^T\rangle$:

$$\mathbf{c} = \mathbf{F}^{-1}\mathbf{b} \ . \tag{5}$$

We assume that the number N of observations y_n is large, so that $\delta\mathbf{b} = \mathbf{b}_{\text{est}} - \mathbf{b}$ and $\delta\mathbf{F} = \mathbf{F}_{\text{est}} - \mathbf{F}$ are small. Thus, we may approximate $\mathbf{F}_{\text{est}}^{-1}$ by $\mathbf{F}^{-1} - \mathbf{F}^{-1}\delta\mathbf{F}\mathbf{F}^{-1}$. We now calculate $\delta\mathbf{c} = \mathbf{c}_{\text{est}} - \mathbf{c}$, the difference between the estimated value and the exact value of the autoregression coefficients. We retain only terms up to first order:

$$\begin{aligned}
\delta\mathbf{c} &= \mathbf{F}_{\text{est}}^{-1}\mathbf{b}_{\text{est}} - \mathbf{F}^{-1}\mathbf{b} \\
&\simeq (\mathbf{F}^{-1} - \mathbf{F}^{-1}\delta\mathbf{F}\mathbf{F}^{-1})\cdot(\mathbf{b} + \delta\mathbf{b}) - \mathbf{F}^{-1}\mathbf{b} \\
&\simeq \mathbf{F}^{-1}\delta\mathbf{b} - \mathbf{F}^{-1}\delta\mathbf{F}\mathbf{F}^{-1}\mathbf{b} \\
&= \mathbf{F}^{-1}(\delta\mathbf{b} - \delta\mathbf{F}\mathbf{c}) \\
&= \mathbf{F}^{-1}(\mathbf{b}_{\text{est}} - \mathbf{F}_{\text{est}}\mathbf{c}) \ ,
\end{aligned} \tag{6}$$

where we have used Eqs. 4 and 5 in the final two steps.

The NLAR model Eq. 2 provides an alternative expression for $\mathbf{b}_{\text{est}}$ in terms of the innovations x_n:

$$\begin{aligned}
\mathbf{b}_{\text{est}} &= \langle \mathbf{f}(Y_n)y_n\rangle_{\text{est}} \\
&= \langle \mathbf{f}(Y_n)\,[\mathbf{f}(Y_n)^T\mathbf{c} + x_n]\rangle_{\text{est}} \\
&= \mathbf{F}_{\text{est}}\mathbf{c} + \langle \mathbf{f}(Y_n)x_n\rangle_{\text{est}} \ .
\end{aligned} \tag{7}$$

Combining Eqs. 6 and 7 yields a simple expression for $\delta\mathbf{c}$ in terms of the innovations x_n and exact model parameters:

$$\delta\mathbf{c} = \mathbf{F}^{-1}\langle \mathbf{f}(Y_n)x_n\rangle_{\text{est}} \ . \tag{8}$$

We now proceed to calculate the covariance matrix $\mathbf{C} = \langle \delta\mathbf{c}\delta\mathbf{c}^T\rangle$:

$$\begin{aligned}
\mathbf{C} &= \left\langle \left(\mathbf{F}^{-1}\frac{1}{N}\sum_{n=1}^{N}\mathbf{f}(Y_n)x_n\right)\cdot\left(\mathbf{F}^{-1}\frac{1}{N}\sum_{m=1}^{N}\mathbf{f}(Y_m)x_m\right)^T\right\rangle \\
&= \mathbf{F}^{-1}\frac{1}{N^2}\left\langle \sum_{n,m=1}^{N}\mathbf{f}(Y_n)\mathbf{f}(Y_m)^T x_n x_m\right\rangle \mathbf{F}^{-1} \\
&= \frac{1}{N^2}\mathbf{F}^{-1}(NU)\langle \mathbf{f}(Y_n)\mathbf{f}(Y_m)^T\rangle\mathbf{F}^{-1} \\
&= \frac{U}{N}\mathbf{F}^{-1} \ .
\end{aligned} \tag{9}$$

In this calculation, the crucial step is recognizing that x_n is independent of $\mathbf{f}(Y_n)$, and if $n > m$, x_n is also independent of x_m and $\mathbf{f}(Y_m)$. Thus terms with $n > m$ (as well as terms with $m > n$) drop out. The terms with $n = m$ do not drop out, and we use the fact that $\langle x_n x_n \rangle = U$. The matrix $\mathbf{C}$ represents the variances and covariances of the estimates $\mathbf{c}_{\mathrm{est}}$ about their mean value $\mathbf{c}$, and thus Eq. 9 provides an index of reliability for these values.

We wish to use this framework to test the incremental significance of adding one term $f_J(Y_n)$ to the AR model Eq. 2. At this stage, it is convenient to assume that the additional AR term $f_J(Y_n)$ is orthogonal to the rest (11). This assumption of orthogonality is not a restrictive one. If the original choice f_J is not orthogonal to the previous f_j $(j = 1, \ldots, J - 1)$, then it may be replaced by an appropriate linear combination $f'_J = f_J - \Sigma a_j f_j$, which is orthogonal to the previous f_j. This orthogonalization amounts to a reorganization of the AR model in which the coefficients $c_{j,\mathrm{est}}$ of the first $J-1$ terms f_j have been changed by a_j, but the coefficient of $c_{J,\mathrm{est}}$ of f_J remains unchanged.

With the assumption that the final term f_J is orthogonal to the previous ones, the symmetric matrix $\mathbf{F}$ assumes a block form, with $F_{j,J} = 0$ and $F_{J,j} = 0$ for $j \neq J$, and $F_{J,J} = \langle [f_J(Y_n)]^2 \rangle$. Its inverse $\mathbf{F}^{-1}$ must similarly be in block form, with $F^{-1}_{J,J} = 1/F_{J,J}$. Consequently, it follows from Eq. 9 that the variance $C_{J,J} = \langle \delta c_J^2 \rangle$ of the deviation $\delta c_J = c_{J\mathrm{est}} - c_J$ is given by

$$\langle \delta c_J^2 \rangle = \frac{U}{NF_{J,J}} \ . \tag{10}$$

In the limit of large N, $\mathbf{F}$ may be approximated by $\mathbf{F}_{\mathrm{est}}$, and the variance U of the driving terms may be replaced by $V = \langle x_n^2 \rangle_{\mathrm{est}}$. Both of these approximations incur errors which are on the order of $1/N$. This error in the reliability estimate is also the extent to which the empirical orthogonalization may differ from orthogonalization with respect to the entire ensemble. Within this uncertainty, an estimate of the reliability of c_J from empirical quantities is

$$\langle \delta c_J^2 \rangle = \frac{V}{NF_{J,J,\mathrm{est}}} \ . \tag{11}$$

Note that in deriving Eq. 11, we did not need to assume that the innovations x_n were Gaussian-distributed; we only needed to use the fact that they were independent and of variance V. If the x_n are indeed Gaussian-distributed and the autoregression rule is linear, then the least-squares estimates provided by the Yule–Walker Eqs. 4 become maximum-likelihood estimates. This may be converted to a confidence interval if the x_n are assumed to be distributed in a Gaussian fashion: For large values of N, $c_{J,\mathrm{est}} \pm t_{\mathrm{crit}}\, [V/(NF_{J,J,\mathrm{est}})]^{1/2}$ represents confidence limits for c_J, where t_{crit} is the critical value of the t-statistic corresponding to the desired level of confidence.

On this basis, we propose a criterion that a measured value of $c_{J,\mathrm{est}}$ reflects an exact value c_J, which is different from zero:

$$|c_{J,\mathrm{est}}| > t_{\mathrm{crit}} \left[\frac{V}{NF_{J,J}} \right]^{1/2} \ . \tag{12}$$

As the above discussion implies, this criterion rigorously corresponds to the statement that the confidence interval for c_J does not include zero, provided that the innovations are Gaussian-distributed and the AR model is linear. However, it also provides a rigorous test for the significance of the coefficient c_J of a lone nonlinear term f_J in an otherwise linear AR model. This is because under the null hypothesis that this coefficient is zero, the model is indeed linear. Once the model is known (or assumed) to include one nonlinear term, then application of the criterion Eq. 12 to determine the significance of a second nonlinear term is only an approximation, since an NLAR model typically requires non-Gaussian distributions of the innovations x_n (e.g., (5)).

The Akaike Criterion

Akaike's minimum information theoretic criterion (1) is commonly used to determine the appropriate number of terms in an LAR model. According to this criterion, the number of terms to be included in the model is the value that minimizes

$$\mathrm{AIC}(J) = N \log V + 2J \ , \tag{13}$$

where V is the residual variance of the J-term AR model.

The coefficient c_J of the final term is thus considered to be significant if $\Delta\mathrm{AIC}(J) = \mathrm{AIC}(J) - \mathrm{AIC}(J-1) < 0$. That is,

$$N \log \frac{V}{V_d} + 2 < 0 \ , \tag{14}$$

where V_d is the residual variance for the best-fitting model with the Jth term f_J omitted. We make the approximation that the incremental decrease in variance due to the Jth term, $\Delta V = V_d - V$, is small in comparison with V. Then, $\log V/(V + \Delta V)$ is well-approximated by $-\Delta V/V$, and Eq. 14 may be rewritten

$$\Delta V > 2 \frac{V}{N} \ . \tag{15}$$

The relationship between the criterion Eqs. 15 and 12 follows from an expression for the residual variance V in the least-squares best-fit AR model of J components:

$$\begin{aligned} V &= \langle x_n^2 \rangle_{\mathrm{est}} \\ &= \langle (y_n - \mathbf{f}(Y_n)^T \mathbf{c})^2 \rangle_{\mathrm{est}} \\ &= \langle y_n^2 \rangle_{\mathrm{est}} - 2\mathbf{b}_{\mathrm{est}}^T \mathbf{c}_{\mathrm{est}} + \mathbf{c}_{\mathrm{est}}^T \mathbf{F} \mathbf{c}_{\mathrm{est}} \ , \\ &= \langle y_n^2 \rangle_{\mathrm{est}} - \mathbf{c}_{\mathrm{est}}^T \mathbf{F}_{\mathrm{est}} \mathbf{c}_{\mathrm{est}} \ , \end{aligned} \tag{16}$$

where the last step made use of Eq. 4 in the form $\mathbf{b}_{\mathrm{est}} = \mathbf{F}_{\mathrm{est}} \mathbf{c}_{\mathrm{est}}$.

The orthogonality assumption for f_J and the assumption that $\mathbf{F}_{\mathrm{est}}$ may be replaced by $\mathbf{F}$ provide a simple expression for $\Delta V = V_d - V$:

$$\Delta V \simeq c_{J,\mathrm{est}}^2 F_{J,J,\mathrm{est}} \ . \tag{17}$$

With this relationship, the Akaike criterion Eq. 15 becomes

$$|c_{J,\mathrm{est}}| > \sqrt{2}\left[\frac{V}{NF_{J,J,\mathrm{est}}}\right]^{1/2} . \tag{18}$$

This equation displays a fundamental similarity between the Akaike criterion Eq. 15 and the criterion Eq. 12 derived above from reliability estimates. The correspondence becomes exact for a value of $t_{\mathrm{crit}} = \sqrt{2}$, which (in the limit of large N) is the critical value for a confidence interval of $p = 0.84$.

The Eqs. 12 and 15 may be rearranged (via Eq. 17) into the common form

$$\frac{N\Delta V}{V} > k , \tag{19}$$

where $k = 2$ for the Akaike criterion Eq. 18 and $(t_{\mathrm{crit}})^2$ for the reliability estimate criterion Eq. 12. It is worthwhile noting that the factor 2 in the Akaike criterion Eq. 15 is a matter of some debate; Leontaritis and Billings (16) argue for the factor of 4, which would (via Eq. 19) correspond to the more traditional confidence interval of $p = 0.95$.

The rationale for the Akaike criterion depends heavily on the assumption of a Gaussian distribution of the innovations, so that a least-squares estimate will coincide with a maximum-likelihood estimate. The error estimation procedure described above properly estimates the covariance of errors in the parameters c_J even when the AR model is nonlinear, without the need to assume that the innovations are distributed in a Gaussian fashion. The correspondence between these criteria thus provides a rationale for extension of the Akaike criterion to NLAR models, since the error-estimate criterion to which it corresponds depends only weakly on the assumptions of linearity and Gaussian distributions.

An Example

Here, we describe an application of the reliability estimate criterion to an NLAR characterization of an ictal EEG discharge. EEG signals acquired with standard clinical techniques (gold cup electrodes filled with electrolyte paste and placed on the scalp according to the 10–20 system) were amplified, filtered (0.3 Hz to 70 Hz), and recorded by a pulse-width modulation technique at 250 Hz with Telefactor (West Conshohocken, PA) telemetry apparatus. This arrangement permitted simultaneous recording of a video image of the patient, so that we could be sure that movement artifact was not present.

The data we analyze (Fig. 1) is a sample of 4.1 s of EEG during a typical run of 3/s spike and wave in an adult with absence ("petit mal") epilepsy (20). As seen in Fig. 1, similar waveforms are present throughout the record, and there is no pattern of temporal evolution. This stationarity of the EEG signal (which is typical of petit mal discharges, but not necessarily of other seizures) is one formal requirement for the autoregressive framework we have discussed above.

Autoregressive analysis was applied to the tracing of channel 7 (Fp1–F7), resampled at 125 Hz. Each data point used for the analysis was the arithmetic mean of two

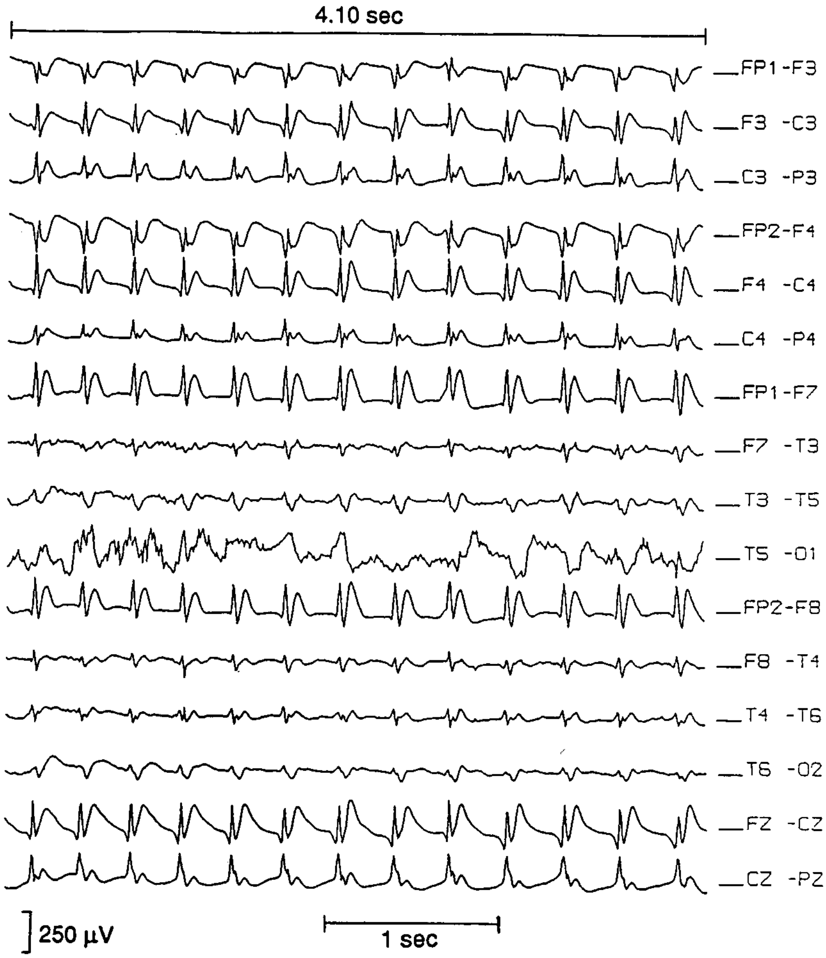


FIGURE 1. EEG recorded in a standard parasagittal montage during a typical 3/s spike and wave seizure discharge.

points obtained from the 250 Hz digitization. This channel was chosen because the spike discharge was large and the record was free of artifact. We used the Yule–Walker Eqs. 4 to estimate coefficients **c** for a model with an offset term $f_0(Y_n) = 1$ augmented by increasing numbers of single-lag terms $f_j(Y_n) = y_{n-j}$. (We included an offset term so that at the next stage of the analysis, the dynamic effect of quadratic terms could be dissociated from their nonzero mean.) The Akaike criterion applied to these modified LAR models yielded a local minimum with $J = 5$ lags which accounted for 81.51% of the variance (Figs. 2a and 2b). After a transient rise in the Akaike criterion value, additional lags (beyond 11) continued to result in further small decreases in the criterion. As seen in Fig. 2c, the minimum of the Akaike criterion is the value of J at which $N\Delta V/V$ (a) exceeds 2 and (b) the subsequent value of $N\Delta V/V$ is less than 2. Beyond 11 lags, most additional terms were associated with values of

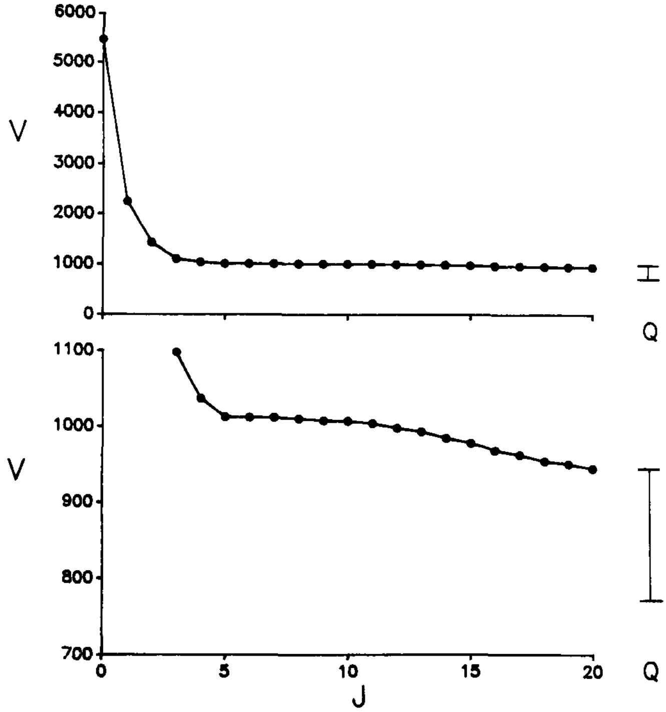


FIGURE 2. (a) Analysis of significance of autoregressive terms. Residual variance *V*.

$n\Delta V/V$ in the range of 2–4, and were, thus, of minimal significance by the Akaike criterion.

The next step in our analysis was to introduce a single quadratic term $f_{jk}(Y_n) = y_{n-j}y_{n-k}$ into an otherwise linear AR model Eq. 2. We were interested in nonlinear terms $y_{n-j}y_{n-k}$ for values of j and k up to 20 (time-lags 8 to 160 ms). As the above analysis shows, the size of the reduction in unexplained variance, due to a specific nonlinear AR term, depends, in part, on how close it is to linear combinations of previously included terms. Thus, if one added nonlinear terms to a 5-lag AR model, one might anticipate that the term $y_{n-8}y_{n-8}$ might contribute more than the term $y_{n-3}y_{n-3}$ merely because 8-lag terms were not present in the *linear* model. To avoid this possible confounding effect, we included linear terms up to lag 20, so that the relative contributions of nonlinear terms over a similar range of lags could be compared. The LAR model truncated at $J = 20$ lags accounted for 82.74% of the variance, which represented only a minimal improvement over the 5-lag model (81.51%).

The Yule–Walker Eqs. 4 for an NLAR model consisting of an offset, 20 linear terms, and a single quadratic term was used to calculate the value of the coefficient c_{jk} for f_{jk} and the resulting reduction in the variance ΔV_{jk}. This analysis was applied separately for all values of j and k in the range from 1-20. The pattern of values of the residual V_{jk} of the model with a single nonlinear term and the corresponding NLAR coefficients c_{jk} are shown in Fig. 3. (The symmetry of the contour map is necessitated by the fact that $f_{jk} = f_{kj}$). These maps are generally similar in structure,

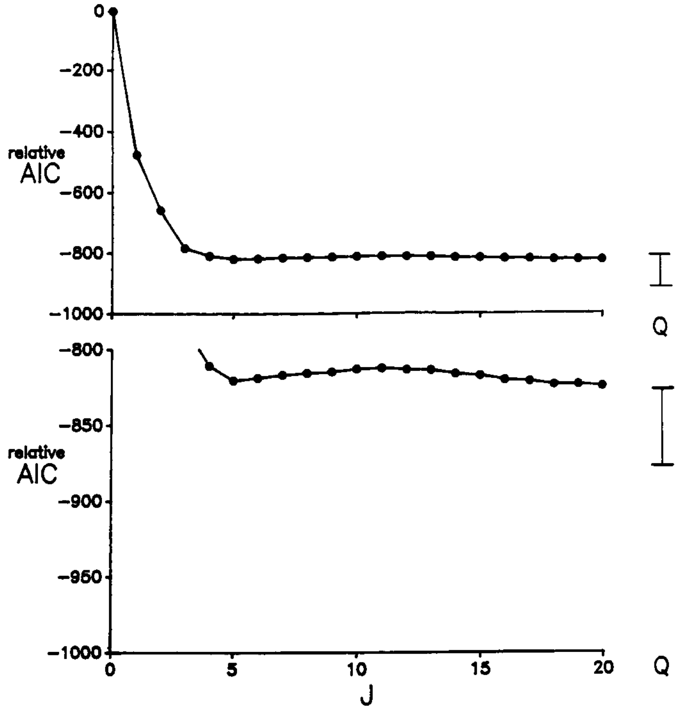


FIGURE 2. (b) Relative Akaike criterion, obtained by subtracting the values provided by Eq. 13 for the number of lags under consideration from the value provided by Eq. 13 with no AR terms except a constant term.

because the matrix **F** in the Yule–Walker Eqs. 4 may always be considered to be in block form, once f_{jk} is orthogonalized. The most prominent features of these contour maps are ridges parallel to the main diagonal, running from time-lags of (4,7) to (8,11), which corresponds to lags of (32 ms, 56 ms) to (64 ms, 88 ms). In addition, there are secondary extrema at (1,8) (corresponding to (8 ms, 64 ms)), at (8,17) (corresponding to (64 ms, 136 ms)), and at (15,15) (corresponding to (120 ms, 120 ms)).

The Akaike criterion indicated a significant improvement of the model from the addition of any single quadratic term. The reduction in the value of the Akaike criterion varied from minimal [0.8 for (j,k) = (9,9)] to substantial [100. for (j,k) = (5,8)] (Fig. 2b). This corresponded to values of $N\Delta V/V$ ranging from 2.8 for (j,k) = (9,9) to 113. for (j,k) = (5,8) (Fig. 2c). As seen in Fig. 3, nearly all of the terms were significant by the stricter choice of $N\Delta V/V > 4$ (16). For the NLAR model with the single term (j,k) = (5,8) included, 85.96% of the variance was accounted for. This amounted to 18.66% of the variance, which remained after 20 linear terms. The variance reduction due to this single term was as much as the reduction in variance due to linear terms of lag 3 through 20.

DISCUSSION

It has often been suggested that the AR formalism might be usefully extended to include nonlinear terms (5,10,12,14,15,24). We find this idea particularly attractive

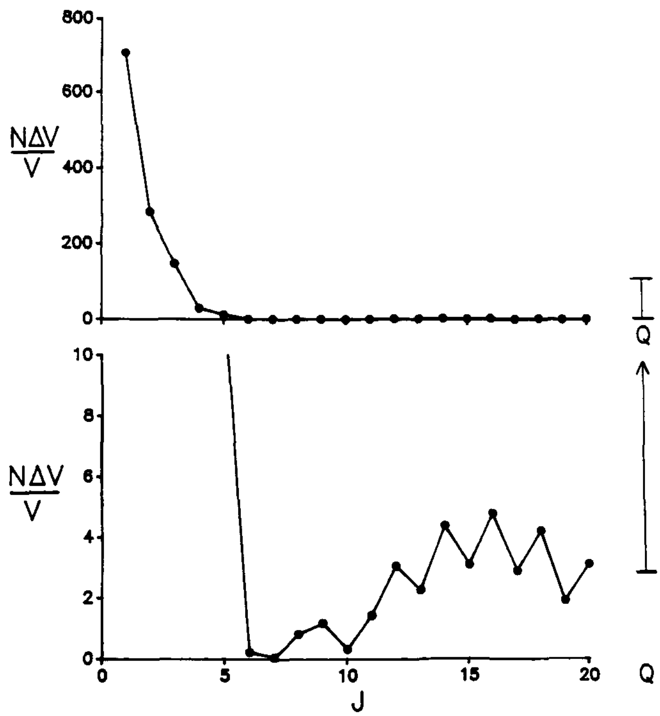
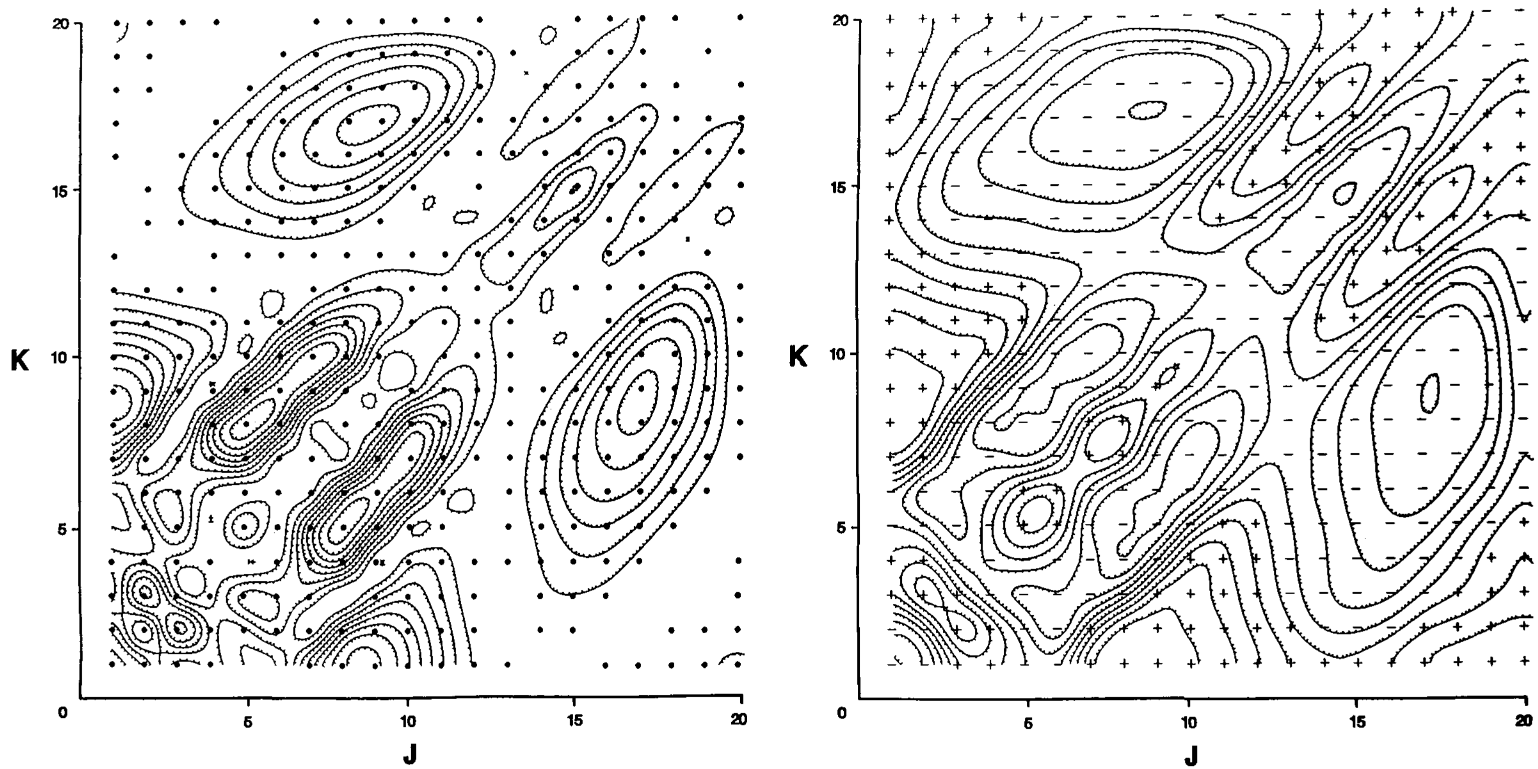


FIGURE 2. (c) $N\Delta V/V$. For linear models, the value of each parameter is plotted as linear AR terms are incrementally added. At the right side of each graph, the range of values obtained for all of the possible quadratic terms is indicated.

in the context of the EEG. The general features of equations thought to underlie EEG dynamics (25) are highly nonlinear. Under normal circumstances, the EEG is the summed result of many relatively independent generators, and thus the resulting signal has very nearly Gaussian behavior (6,17,18,22,23). Thus, the microscopic dynamics bear little relationship to the normal surface EEG. As a consequence, despite the likely nonlinear nature of the underlying dynamics (8,17,25,26), LAR models form an excellent compact description of the normal EEG (1,7). However, during seizure activity, the EEG is dominated by abnormally large populations of synchronized neurons. Such paroxysmal activity characteristically displays a polarity (i.e., sharp transients that are unidirectional), and thus cannot be well-represented by a linear AR model. Indeed, a preliminary report indicates that EEG discharges during frank seizures display features of nonlinear recursion (21). For these reasons, we thought it likely that an NLAR formalism might provide a useful characterization of EEG dynamics during a seizure discharge. We recognize that a detailed correspondence between an NLAR model and a microscopic description of EEG dynamics is not likely within reach. Rather, our goal is to use NLAR models to exclude some models for EEG generation, and perhaps suggest others.

To deduce an NLAR model from data, some means of deciding on the statistical significance of nonlinear terms is necessary. As pointed out above, the Akaike criterion, as originally proposed, cannot be rigorously applied in the nonlinear case. One



(a) (b)

FIGURE 3. (a) Contour maps of the value of the residual V_{jk}; and (b) the coefficient c_{jk} obtained by introducing single quadratic terms $f_{jk}(Y_n) = y_{n-j}y_{n-k}$ into an otherwise linear AR model Eq. 2. Each unit of time represents 8 ms. In (a) each contour line represents 20 μv^2 and the dot at each data point is suppressed at locations at which $N\Delta V_{jk}/V < 4$. In (b) each contour line represents 0.1 μv^{-1} and the "+" or "−" sign at each data point indicates the signature of c_{jk}. In both maps, the tick-marks indicate the downhill side of the contour lines.

approach is to determine an estimate of the reliability of the coefficient of a potential nonlinear term. If this coefficient is not reliably different from zero, then there is no statistical justification for retaining the term. If this coefficient is reliably different from zero, then the null hypothesis (that the term under consideration is not present) may be rejected.

Our analysis is similar to the log-determinant ratio method for identification of structure of NLAR models with moving-average components and accessible inputs, as advanced by Billings and coworkers (3,14–16). In the present setting (a) the model has no moving-average components, (b) the input is inaccessible and, therefore, not an explicit component of any of the AR terms f_j, and (c) we consider the significance of a single nonlinear AR term in an otherwise linear AR model. Because of the structure of the NLAR model and the least-squares fitting procedure, these specializations lead to reliability estimates for the coefficients of orthogonalized AR terms are simply related to the reduction in variance provided by these terms. A criterion essentially identical to Eq. 12 has already been proposed (13); our point here is to emphasize that this criterion is philosophically equivalent to that of Akaike (1), rather than an alternative (cf. (13)).

One problem inherent in autoregressive modelling a signal such as that shown in Fig. 1 is that since the signal is nearly periodic, the set of recent lagged values explores a state-space of low dimension (2). For this reason, one may wonder that the appearance of nonlinear terms in the autoregressive model may be a consequence of the fact that for the trajectory of state-space traversed, nonlinear and linear autoregressive terms may not be distinguishable. Were this the case, however, then the nonlinear components in $\mathbf{f}$ would be linearly dependent (or nearly so) on the linear terms, and they would not lead to a significant reduction in the variance V.

As we have shown, application of this procedure to the EEG recorded during a seizure discharge indicates that an LAR model is improved by the addition of nonlinear terms. This by itself is not surprising, and might well be suggested by visual inspection of the raw data. What is not evident from visual inspection of the EEG tracing is the pattern of significance of the nonlinear interactions, as revealed by the contour map of Fig. 3.

To a first approximation, the contour map of Fig. 3b may be considered to be a second-order Volterra kernel (19) of the nonlinear filter in a hypothetical feedback loop, which determines the current signal value y_n from its previous values. Qualitative features of a second-order Volterra kernel $K_2(\tau_1,\tau_2)$ provide clues to the nature of the transduction they describe. For example, a system consisting of a linear filter with impulse response $G(\tau)$ followed by a static nonlinearity has a second-order kernel of the form

$$K_2(\tau_1,\tau_2) = aG(\tau_1)G(\tau_2) \ . \tag{20}$$

More generally (9), a system consisting of a linear filter with impulse response $G(\tau)$ followed by a static nonlinearity, followed by a second linear filter of impulse response $H(\tau)$, has a second-order kernel of the form

$$K_2(\tau_1,\tau_2) = a\int_0^\infty G(\tau_1-\tau)G(\tau_2-\tau)H(\tau)\,d\tau \ . \tag{21}$$

The contour map of Fig. 3b does not readily fit into the form Eq. 20 or Eq. 21, either of which would typically be dominated by contours running parallel to the axes. Rather, the most striking feature of this map is the valley parallel to the main diagonal, running from time-lags of (32 ms, 56 ms) to (64 ms, 88 ms). This suggests a feedback process driven by a nonlinear interaction of EEG values at times separated by 24 ms. It is interesting to note that this time interval is comparable to the time scale of the spike component of the 3/s discharge. Our statistical analysis demonstrates that this feature is *not* a consequence of the frequencies present in the seizure discharge—were this the case, the reduction in variance, due to these nonlinear terms, would be no greater than that expected for a signal generated by the best-fitting LAR model.

Because our goal is to characterize the dynamics of the EEG (rather than to provide as concise a representation as possible), we elected to retain all linear terms up to 20 lags in the LAR characterization, even though some of them were not significantly different from zero. Had we chosen to omit some of these terms, we might have introduced spurious elements into the contour maps of Fig. 3 for lags corresponding to those omitted from the LAR model.

We emphasize that although the NLAR procedure indicates the presence of nonlinear dynamics, an NLAR model with one or more quadratic terms is unlikely to represent an adequate physiological model for the EEG. In principle, an NLAR model with quadratic terms is likely to be unstable unless the innovations are carefully restricted. The true nature of the nonlinear feedback is unlikely to be polynomial; thresholds and rectifications (24) are likely more appropriate. The analysis shown in Fig. 3 provides a fingerprint of the nonlinear dynamics, which can serve both as a characterization of this EEG discharge and a means to test physiological models, but does not constitute a physiologic model as it stands.

REFERENCES

1. Akaike, H. A new look at the statistical model identification. IEEE Trans. Auto. Control AC-19:716–723; 1974.
2. Babloyantz, A.; Destexhe, A. Low-dimensional chaos in an instance of epilepsy. Proc. Natl. Acad. Sci. 83:3513–1317; 1986.
3. Billings, S.A.; Chen, S. Identification of nonlinear rational systems using a prediction-error estimation algorithm. Int. J. Syst. Sci. 20:467–494; 1989.
4. Box, G.E.P.; Jenkins, G.M. Time series analysis: Forecasting and control. San Francisco: Holden-Day; 1976.
5. Dewald, L.S.; Lewis, P.A.W. A new Laplace second-order autoregressive time-series model-NLAR(2) IEEE Trans. Information Theory IT-31:645–651; 1985.
6. Elul, R. Gaussian behavior of the electroencephalogram: Changes during performance of mental task. Science 164:328–331; 1969.
7. Gersh, W.; Yonemoto, J. Parametric time series models for multivariate EEG analysis. Comp. Biomed. Res. 10:113–125; 1977.
8. Kawahara, T. Coupled Van der Pol oscillators—A model of excitatory and inhibitory neural interactions. Biol. Cybern. 39:37–43; 1980.
9. Korenberg, M.J. Cross-correlation analysis of neural cascades. Proc. 10th Ann. Rocky Mountain Bioeng. Symp., 47–51; 1973.
10. Korenberg, M. Orthogonal identification of nonlinear difference equation models. Proc. 28th Midwest Symp. Circuit Syst. 1:90–95; 1985.
11. Korenberg, M. Functional expansions, parallel cascades, and nonlinear difference equations. In Marmarelis, V.Z., ed. Advanced methods of physiological system modelling, Vol. 1, Los Angeles: University of Southern California; 1987:pp. 221–240.

12. Korenberg, M. Exact orthogonal estimation of kernels with biological applications. Proc. IEEE Conf. Biomed. Technol. 27-30; 1987.
13. Korenberg, M. Fast orthogonal algorithms for nonlinear system identification and time-series analysis. In: Marmarelis, V.Z., ed. Advanced methods of physiological system modelling, Vol. 2, New York: Plenum Press; 1989:pp. 160-177.
14. Leontaritis, I.J.; Billings, S.A. Input-output parametric models for non-linear systems. I. Deterministic non-linear systems. Int. J. Control 41:303-328; 1985.
15. Leontaritis, I.J.; Billings, S.A. Input-output parametric models for non-linear systems. II. Stochastic non-linear systems. Int. J. Control 41:329-344; 1985.
16. Leontaritis, I.J.; Billings, S.A. Model selection and validation methods for non-linear systems. Int. J. Control 45:311-341; 1987.
17. Lopes da Silva, F.H.; Hoeks, A.; Smits, H.; Zetterberg, L.H. Model of brain rhythmic activity: The alpha-rhythm of the thalamus. Kybernetik 15:27-37; 1974.
18. Lopes da Silva, F.H.; van Lierop, T.H.M.T.; Schrijer, C.F.; Storm van Leeuwen, W. Organization of thalamic and cortical alpha rhythms: Spectra and coherences. Electroencephalog. Clin. Neurophysiol. 35:627-639; 1973.
19. Marmarelis, P.Z.; Marmarelis, V.Z. Analysis of physiological systems: The white-noise approach. New York: Plenum Press; 1978.
20. Niedermeyer, E. Epileptic seizure disorders. In: Niedermeyer, E.; Lopes da Silva, F., eds. Electroencephalography: Basic principles, clinical applications, and related fields. Baltimore: Urban and Schwartzenberg; 1983:pp. 339-428.
21. Saltzberg, B.; Burton, W.D., Jr. Nonlinear filters for tracking chaos in neurobiological time series. In: Marmarelis, V.Z., ed. Advanced methods of physiological system modelling, New York: Plenum Press; 1989:pp. 201-213.
22. Siegel, A. Stochastic aspects of the generation of the electroencephalogram. J. Theor. Biol. 92:317-339; 1981.
23. Storm van Leeuwen, W. The alpha rhythm. In: Cobb, W.A.; Van Duijn, H., eds. Contemporary clinical neurophysiology (EEG suppl. No. 34). Amsterdam: Elsevier; 1978:pp. 1-7.
24. Tong, H. Threshold models in non-linear time series analysis. Lecture notes in statistics, vol. 21. D. Brillinger *et al.*, eds. New York: Springer-Verlag; 1983.
25. Wilson, H.R.; Cowan, J.D. A mathematical theory of the functional dynamics of cortical and thalamic nervous tissue. Kybernetik 13:55-80; 1973.
26. Wright, J.J.; Kydd, R.R.; Lees, G.J. State-changes in the brain viewed as linear steady-states and non-linear transitions between steady-states. Biol Cybern. 53:11-17; 1985.
27. Yule, G.U. On a method of investigating periodicities in disturbed series with special reference to Wolfer's sunspot numbers. Phil. Trans. Roy. Soc. Lond. A 226:267-298; 1927.