A COUNTING PROBLEM ARISING IN NONLINEAR SYSTEMS ANALYSIS

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ABSTRACT. Let $h_k(n,r)$ be the number of ways that r objects out of a limitless supply of objects, distinguishable only by their n colors, can be distributed onto k points of a circle, given that objects of each color occur on at most one point, and counting only once arrangements that differ only by cyclic permutation of the points. The numbers $h_2(n,r)$ arise naturally in the study of the response of a nonlinear transducer to a sum of incommensurate sinusoids. For the general $h_k(n,r)$, a recurrence relation, a generating function, and explicit expressions are determined.

Introduction.

In this note, we state and solve a counting problem that arises in the theory of nonlinear systems analysis. The recurrence relation that generates the solutions to the counting problem is the "square" functional equation of Stanton and Cowan [4], but has different initial conditions. The specific counting problem of interest has a natural generalization, which is treated with no additional difficulties.

The counting problem arises when one considers the response of a nonlinear transducer to a signal that is a superposition of n sinusoids of incommensurate frequencies $\{\alpha_j\}$. Such a procedure forms the core of a practical method for the analysis of nonlinear neural transducers [5]. A nonlinear transducer's response to a sum of sinusoids may be represented as a trigonometric sum involving not only the n input frequencies, but also their sums and differences [1]. It is natural to inquire how many distinct (positive) frequencies of order r may occur in this expansion. That is, how many positive numbers may be written in the form

$$\sum_j \varepsilon_j c_j \alpha_j, \quad \text{where the } \{c_j\} \text{ are integers, } \varepsilon_j = \pm 1, \text{ and } \sum_j c_j = r.$$

The number of such expressions is equal to the number of ways of choosing r numbers (possibly with repetition) out of the set of n numbers $\{\alpha_j\}$, and assigning a signature ε_j to each of the α_j's that are chosen at

UTILITAS MATHEMATICA Vol. 16 (1979), pp. 151-156.

least once. Each pair of arrangements related by inversion of all signatures is counted only once, for in only one instance will the sum be positive.

Abstractly, one is asked to choose r samples out of a supply of n distinguishable kinds of objects, and to sort them into two piles, such that each kind of object appears in at most one pile, and arrangements that differ only by cyclic permutation of the piles are considered identical. The number of such arrangements is just as easy to determine for k piles as it is for two, and will be denoted $h_k(n,r)$. This constitutes the counting problem explored below.

The enumeration $h_k(n,r)$ can be accomplished by a direct argument that leads to equation (12) below. However, the argument presented below, which is based on a recurrence relation and the use of a generating function, shows the relationship of the numbers $h_k(n,r)$ to a tableau of Stanton and Cowan [4], and generalizes a well-known argument of Feller [3].

A Counting Problem.

We seek $h_k(n,r)$, the number of ways that r objects out of a limitless supply of objects distinguishable only by their n colors may be distributed on k equally spaced points of a circle, such that objects of each color occur on at most one point. Arrangements that differ only by rotation of the circle are considered identical. Note that $h_1(n,r)$ is the number of terms in the multinomial expansion of an n-term expression raised to the r^{th} power, and is $\binom{n+r-1}{r}$. As pointed out above, the number $h_2(n,r)$ arises in the theory of the response of a nonlinear system to a sum of incommensurate sinusoids [1]: it is the number of possible r^{th} order combination frequencies in the transducer's response to an input sum of n incommensurate sinusoids.

To evaluate $h_k(n,r)$, we first prove a recurrence relation:

$$(1) \qquad h_k(n,r) = h_k(n-1,r) + h_k(n,r-1) + (k-1)\, h_k(n-1,r-1)$$

for $n,r \geq 2$. The relation (1) is a generalization of the "square" recurrence relation [4]

$$g(n,r) = g(n-1,r) + g(n,r-1) + g(n-1,r-1) \ .$$

The latter relation has been generalized in a different manner by Carlitz [2]:

$$A(n,r) = p^r A(n-1,r) + p^n A(n,r-1) + A(n-1,r-1) .$$

The relationship of Carlitz's recurrence formula and equation (1) is demonstrated by taking $p = k-1$ and $B(n,r) = p^{nr} h_k(n,r)$:

$$B(n,r) = p^r B(n-1,r) + p^n B(n,r-1) + p^{n+r} B(n-1,r-1) .$$

To prove the present recurrence relation (1), we use a generalization of a well-known argument [3] that was originally applied in the case of $k = 1$. To each solution of the counting problem, we associate a string of the symbols dagger (†), slash (/),and asterisk (*) with the following interpretation.

Begin at some particular point on the circle, with some particular color. Each occurrence of the "†" means to place one object of the current color at the current vertex. Each occurrence of the "/" means to change to the next color. Each occurrence of the "*" means to move clockwise to the next point. Thus, to obtain a unique correspondence of strings of symbols to the counting problem, the following rules must be obeyed:

(i) there must be exactly n †'s;

(ii) there must be exactly $r-1$ /'s;

(iii) *'s can occur only in clumps of $k-1$ or less;

(iv) clumps of *'s must occur immediately following a †, and immediately preceding a /;

(v) there must be a † following the final *.

Now we reduce any string corresponding to $h_k(n,r)$ by removing initial characters as follows. The initial character must be a † or /, by rule (iv). If it is a /, remove it and obtain a valid string corresponding to $h_k(n,r-1)$. If the initial character is a †, remove it unless it is followed immediately by a * (this would cause the shortened string to break rule (iv)), and obtain a valid string corresponding to $h_k(n-1,r)$. If the initial character is a † followed by a clump of *'s, remove the symbol †, the clump of *'s, and the following /. This results in a valid string corresponding to

$h_k(n-1,r-1)$. It is easy to verify that this procedure yields each of the $h_k(n,r-1)$-strings exactly once, each of the $h_k(n-1,r)$-strings exactly once, and each of the $h_k(n-1,r-1)$-strings exactly $k-1$ times (by rule (iii)). This yields the required decomposition (1) of $h_k(n,r)$.

An expression for the generating function

$H_k(x,y) \equiv \sum_{n,r=1}^{\infty} x^n y^r h_k(n,r)$ can easily be deduced from the recurrence relation (1). The requisite initial values $h_k(1,r) = 1$ and $h_k(n,1) = n$ are easily verified from the original counting problem. Thus

$$H_k(x,y) = \frac{xy}{1-x}\left(\frac{1}{1-x-y-(k-1)xy}\right). \tag{2}$$

In order to obtain explicit expressions for $h_k(n,r)$, we introduce a new tableau $g_k(n,r)$ having the same recurrence relation (1) as $h_k(n,r)$ but initial values $g_k(0,r) = g_k(n,0) = 1$. The special case $k = 1$ yields the binomial coefficients, and the case $k = 2$ has been studied previously [4]. The generating function for the g-tableaux follows from a calculation similar to that of equation (2):

$$G_k(x,y) \equiv \sum_{n,r=0}^{\infty} g_k(n,r)x^n y^r = \frac{1}{1-x-y-(k-1)xy}. \tag{3}$$

Relations among $g_k(n,r)$ and $h_k(n,r)$ may be derived from the generating functions. For example,

$$g_k(n,r) = h_k(n+1,r+1) - h_k(n,r+1) \quad \text{for } n,r \geq 1, \tag{4}$$

$$h_k(n+1,r+1) = \frac{1}{k}[(k-1)g_k(n,r) + g_k(n,r+1)] \quad \text{for } n,r \geq 1, \tag{5}$$

$$\sum_{p=1}^{r} h_k(n,p) = \frac{1}{k}[g_k(n,r)-1], \quad \text{and} \tag{6}$$

$$r\, h_k(n,r) = n\, h_k(r,n). \tag{7}$$

These identities follow from equating terms in

$$(1-x)H_k(x,y) = xy\, G_k(x,y) ,$$

$$H_k(x,y) + \frac{x}{k(1-x)} = \frac{x}{k}[(k-1)y+1]G_k(x,y) ,$$

$$\frac{1}{1-y} H_k(x,y) = \frac{1}{k}[G_k(x,y)-G_0(x,y)] , \quad \text{and}$$

$$\frac{1}{x}\frac{\partial}{\partial y} H_k(x,y) = \frac{1}{y}\frac{\partial}{\partial x} H_k(y,x) = [G_k(x,y)]^2 .$$

Explicit Expressions.

The generating function $G_k(x,y)$ may be expanded to obtain explicit expressions for $g_k(n,r)$. For example,

$$G_k(x,y) = \sum_{m,a,b} x^a y^b [(k-1)xy]^{m-a-b} \frac{m!}{a!b!(m-a-b)!} .$$

Taking $m = r + a$, $b = m - n$, shows that

$$g_k(n,r) = \sum_a (k-1)^{n-a} \binom{n}{a}\binom{r+a}{n} . \tag{8}$$

Equation (8) generalizes the $(k = 2)$-result of Stanton and Cowan [4]. An alternate form of (8) may be obtained by standard manipulations:

$$g_k(n,r) = \sum_a (k-1)^{n-a} \binom{n}{a} \sum_b \binom{r}{b}\binom{a}{n-b} = \sum_b \binom{r}{b}\binom{n}{b}k^b . \tag{9}$$

Equation (9) generalizes Lemma 3 of [4].

Several equivalent expressions for $h_k(n,r)$ may now be derived by elementary combinatorial manipulations beginning with relations (8) or (9) and using (4), (5), or (6).

$$h_k(n,r) = \frac{1}{k} \sum_a (k-1)^a \binom{n}{a}\binom{n+r-a-1}{n-1} , \tag{10}$$

$$h_k(n,r) = \sum_a (k-1)^a \binom{r-1}{a}\binom{n+r-a-1}{r} , \tag{11}$$

$$h_k(n,r) = \sum_a k^a \binom{n}{a+1}\binom{r-1}{a} . \tag{12}$$

Acknowledgements. The author thanks John Riordan for his encouragement and helpful suggestions. The work was supported, in part, by NIH Grants EY188 and EY1428.

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Received April 6, 1979.