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Temporal impulse responses from flicker sensitivities: causality, linearity, and amplitude data do not determine phase

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A recent paper [J. Opt. Soc. Am. A **4,** 1130 (1987)] advances a method for the determination of an impulse response from amplitudes of a measured amplitude spectrum. The crucial step in this derivation is the assumption that if a transfer function is analytic and causal, then so is its logarithm. This assumption is tantamount to that of minimum phase and is not justified, both in principle and in practice. In the absence of additional information or assumptions, every amplitude spectrum is consistent with a multiplicity of phase spectra.

INTRODUCTION

Linear systems theory permits the characterization of a transducer either by its impulse response $r(t)$ or, equivalently, by its transfer function $R(\omega)$. Some measurements provide estimates of the impulse-response function, while others provide estimates of the transfer function. Psychophysical measurements with sinusoidal stimulation typically estimate only the amplitude of the transfer function, $|R(\omega)|$. As reviewed by Stork and Falk,[1] relationships between $r(t)$ and $R(\omega)$ are of great interest for testing a variety of psychophysical models.

Stork and Falk[1] propose a method for reconstruction of the complete transfer function $R(\omega)$ from amplitude data alone, without any assumptions other than linearity and causality. The purposes of this Communication are to identify a hidden assumption in their analysis and to remind the reader that phase data are *not* determined by amplitude data.

THEORY

The impulse response $r(t)$ is the response of a transducer to an infinitesimally narrow pulse input at time zero. The transfer function $R(\omega)$ is a complex number that expresses the amplitude and the phase of the transducer's response to a unit sinusoidal input of frequency ω. These quantities are Fourier-transform pairs. In the notation of Stork and Falk,[1]

$$R(\omega) = \int_{-\infty}^{\infty} r(t)e^{i\omega t}\mathrm{d}t \tag{1}$$

and

$$r(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} R(\omega)e^{-i\omega t}\mathrm{d}\omega. \tag{2}$$

Three conditions characterize any frequency-domain function $R(\omega)$ that represents the temporal properties of a physical transducer: (i) $R(\omega)$ must be analytic (i.e., have no singularities) in the upper half of the complex ω plane (here called the causal half-plane). This corresponds to the condition that $r(t) = 0$ for negative t; i.e., $r(t)$ is causal. (ii) $R(-\bar{\omega}) = \bar{R}(\omega)$, where $\bar{\omega}$ denotes the complex conjugate of ω. This corresponds to the condition that $r(t)$ is real valued. (iii) $|R(\omega)|$ approaches zero sufficiently rapidly for large positive or negative real values of ω. The rate of approach is typically taken to be $1/|\omega|$.[1] The reader is referred to standard references (e.g., Ref. 2) for further details.

Analyticity of $R(\omega)$ in the causal half-plane permits determination of the imaginary part of $R(\omega)$ from its real part (the Kramers–Krönig relations[3]). Stork and Falk[1] attempt to reduce the phase-from-amplitude problem to the imaginary-from-real-part problem by taking the logarithm of $R(\omega)$. The problem with their analysis is that the function $\log[R(\omega)]$ is not guaranteed to be analytic in the causal half-plane, even if $R(\omega)$ is analytic [see the assertion following their Eq. (7)]. Failure of analyticity occurs if $R(\omega) = 0$ at any point in the causal half-plane, since $\log(0)$ is not defined.

Transfer functions with no zeros in the causal half-plane are known as minimum-phase transducers.[4] Only for such transducers does the algorithm of Stork and Falk[1] provide the correct phase reconstruction (other than the ambiguity of a pure delay term $e^{i\omega D}$).

DISCUSSION

Minimum-phase transducers are a special subset of transducers. Every minimum-phase transducer belongs to an infinite set of causal transducers with identical amplitudes but different phases. Conversely, every causal transfer function can be decomposed into a product of a delay term, a series of stereotyped unity-gain terms, and a minimum-phase term. These and other aspects of the minimum-phase property are reviewed below.

Consider the function

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$$Z(\omega, u) = \frac{\omega - u}{\omega - \bar{u}} \tag{3}$$

for a complex number u in the causal half-plane. This function is constructed so that $Z(u, u) = 0$, and

$$|Z(\omega, u)| = 1 \tag{4}$$

for all real values of ω. A transfer function $Q(\omega)$ that does *not* have a zero in the causal half-plane may be coverted into transfer functions

$$R_{i\gamma}(\omega) = Q(\omega)Z(\omega, i\gamma) \tag{5}$$

and

$$R_{\alpha+i\beta}(\omega) = Q(\omega)Z(\omega, \alpha + i\beta)Z(\omega, -\alpha + i\beta), \tag{6}$$

which share the same amplitudes but have at least one zero in the causal half-plane located at $u = i\gamma$ or $u = \alpha + i\beta$.

To see that $R_{i\gamma}(\omega)$ and $R_{\alpha+i\beta}(\omega)$ indeed represent physical transducers if $Q(\omega)$ represents a physical transducer, observe that conditions (i)–(iii) above hold for $R_u(\omega)$ whenever they hold for the original transfer function $Q(\omega)$. Inclusion of the second factor of Z for values of u that are not pure imaginary [Eq. (6)] is required to preserve the symmetry condition (ii).

In an analogous manner, any transfer function $R(\omega)$ that has zeros in the causal half-plane may always be decomposed into a product of terms of the form of Eq. (3) and another transduction $Q(\omega)$ that does not have a zero in the causal half-plane. [The existence of pairs of zeros $\alpha + i\beta$ and $-\alpha + i\beta$ as required by Eq. (6) is guaranteed by the symmetry condition (ii).] This decomposition may be formalized as

$$R(\omega) = e^{i\omega D}Q(\omega) \prod_j Z(\omega, u_j). \tag{7}$$

The delay D accounts for any absolute latency component in $r(t)$. That is, it is the largest value for which $r(t + D)$ is causal.

Equation (7) provides a canonical decomposition of a non-minimum-phase transducer into a pure delay, one or more unity-gain terms [Eq. (3)], and a minimum-phase transducer $Q(\omega)$ for which the phase-reconstruction algorithm is valid.

In general, transducers made up of simple RC-stage low-pass filters, diffusion stages, and serial and feedback combinations of such stages are indeed minimum phase. However, parallel combinations of such filters are typically not minimum phase. Indeed, the above construction of prototypical non-minimum-phase transducers leads to filters that are parallel combinations of simple RC stages and the identity transducer. For pure imaginary values $u = i\gamma$,

$$Z(\omega, i\gamma) = 1 - \frac{2}{1 + \dfrac{\omega}{i\gamma}}. \tag{8}$$

The existence of non-minimum-phase transducers is not merely a theoretical concern. When phases as well as amplitudes are measured (typically in a single-unit experiment), non-minimum-phase transductions are often identified. One well-known example of a non-minimum-phase visual transduction is lateral inhibition in the *Limulus*.[5]

A minimum-phase transducer always has a smaller first moment than a corresponding non-minimum-phase transducer with identical amplitudes.[4] This is important, since the first moment corresponds to the effective integration time of the transducer for low-frequency inputs. Thus a minimum-phase reconstruction algorithm potentially underestimates the temporal integration time of the transducer under study.

A second qualitative difference between a minimum-phase transfer function $Q(\omega)$ and a corresponding non-minimum-phase transfer function $R(\omega)$ is that the impulse response $r(t)$ corresponding to $R(\omega)$ tends to have more zero crossings than the impulse response $q(t)$ that corresponds to the minimum-phase transfer function $Q(\omega)$.

The algorithm of Stork and Falk[1] provides a canonical set of phase data (the minimum-phase set) consistent with a measured amplitude spectrum. However, this set of phases is not guaranteed to correspond to the phase spectrum of the transducer under study; it is only one of a continuum of possibilities. The minimum-phase set of phase data corresponds to a transfer function that has no zeros in the causal half-plane. Other sets of phase data equally consistent with the measured amplitude spectra correspond to transfer functions with one or more zeros in the causal half-plane. These phase sets reconstruct impulse responses whose first moment (apparent integration time) is larger than that of the impulse response derived from the minimum-phase set of phases.

It is possible to derive phase data from amplitude data only if some additional assumption (e.g., minimum-phase or a particular model) is made.

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