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Models for preattentive texture discrimination: Fourier analysis and local feature processing in a unified framework

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Abstract—Spatial frequency analysis and local feature analysis may be considered to be examples of a class of models for texture discrimination. In this theoretical framework, texture discrimination relies on differences in the distribution of responses generated in linear receptive fields placed randomly on the texture. If the set of receptive fields is taken to be a collection of gratings, spatial-frequency analysis is recovered. If the set of receptive-fields is taken to be a collection of local feature templates, a corresponding local-feature model is recovered.

In order to test such models, it is necessary to construct distinct texture pairs that elicit similar distributions of responses for all of the postulated receptive field profiles: the model prediction is that such textures are not discriminable. A method is provided for construction of such textures which test generic models within this framework. This framework includes not only strict Fourier analysis, but also models which postulate a collection of arbitrarily-shaped local feature detectors, and models which postulate both Fourier analysis and local feature detection.

INTRODUCTION

The segregation of a region of visual space from its surrounding milieu is an important part of early vision. One hallmark of this process is that it is a parallel one, independent of focused attention (Bergen and Julesz, 1983).

Early texture-discrimination studies (Julesz, 1962) suggested that this segregation process is driven by differences in the second-order correlation statistics of textures, or, equivalently, by differences in their spatial frequency spectrum. A key difficulty in *testing* this hypothesis was the need to construct rich classes of textures which shared second-order correlation statistics, yet differed in their higher-order correlation structure. Initial studies with a restricted class of such textures (Rosenblatt and Slepian, 1962) supported the second-order hypothesis. However, more recently, many examples in which differences in higher-order statistics support texture segregation have been found (Caelli and Julesz, 1978; Caelli *et al.*, 1978; Julesz *et al.*, 1978; Victor and Brodie, 1978; Gagalowitz, 1981; Gagalowitz and Ma, 1985). One interpretation of these findings is that textures are discriminable if they posses significantly different densities of local features ('textons'; Bergen and Julesz, 1983).

However, despite these exceptions, spatial-frequency analysis clearly plays a fundamental role in vision, both at the level of individual visual neurons (Movshon *et al.*, 1978; DeValois *et al.*, 1985) and at the level of psychophysics (Campbell and Green, 1965; Campbell and Robson, 1968). Our goal here is to express the spatial-frequency viewpoint in a manner in which it may naturally be extended to include processing of one or more local feature. We then consider how such theories may be tested.

RESULTS

Overview

We begin with the motivation of the present approach. Next, we define iso-(R^*, ∞) and iso-(R^*, p) textures, and state the main results. We introduce the characteristic function of a texture, which is a convenient descriptor of a texture's statistics. We provide algorithms for the construction of iso-(R^*, ∞) and iso-(R^*, p) textures and examples of the application of these algorithms. We then prove that for sinusoidal receptive fields, iso-$(R^*, 2)$ textures are necessarily iso-(R^*, ∞) textures.

Spatial frequency analysis revisited. Let us consider the spatial-frequency model for the computations underlying preattentive texture discrimination. In this model, two textures are discriminable if (and only if) their spatial frequency contents differ sufficiently.

The spatial frequency content, or spatial power spectrum, may be viewed as a description of the outputs of a collection of idealized receptive fields. Consider a receptive field R_f which sums light linearly according to a spatial weighting which is a sinusoid of spatial frequency f. For technical reasons, we assume that the spatial extent of the receptive field, L, is large in comparison with the spatial period $1/f$, and that the spatial weighting is zero beyond the distance L. For each placement of this receptive field on the texture, the output of R_f is proportional to an estimate of the Fourier component of a local region of the texture at the frequency f. The size (and indeed the signature) of this quantity is dependent on the precise placement of the receptive field R_f on the texture. However, the average value of the *square* of this output is proportional to the value of the spatial frequency spectrum at the frequency f. Thus, the value of the spatial frequency spectrum of a texture at the frequency f is proportional to the second moment (mean-square) of the distribution of responses of sinusoidal receptive fields placed at random on the texture.

Some definitions. Spatial frequency content alone does not account for texture discrimination (Caelli and Julesz, 1978; Caelli *et al.*, 1978; Julesz *et al.*, 1978). Thus, more information is available for texture-discrimination than the second moment of the outputs of sinusoidal receptive fields. We are, therefore, led to consider both higher moments, and non-sinusoidal receptive fields.

We consider a set R^* of arbitrary receptive fields. (The set of sinusoidal profiles will be denoted by R^{Fou}.) Members of a general receptive field set R^* will be denoted by R or R_α. As before, we assume that the receptive fields are linear. That is, the response of a receptive field R_α to a texture stimulus A is the integral of the product of the intensity profile of R_α and the luminance profile of the texture A.

The response of a receptive field R_α to a texture A depends not only on R_α and A, but also on the precise position of R_α within A. Placement of examples of R_α at random positions on A thus produces a distribution of responses. Now consider two textures, A_1 and A_2. For each receptive field R_α in R^*, there are two distributions: one corresponding to responses elicited by placement of R_α on A_1, and one corresponding to responses elicited by placement of R_α on A_2. If, for all receptive field types R_α in R^*, the moments of these distributions match up to some order p, we say that A_1 and A_2 are iso-(R^*,p) pairs. If, for all receptive field types R_α in R^*, the distributions are identical (i.e. their moments match at *all* orders), we say that A_1 and A_2 are iso-(R^*,∞) pairs.

A class of models for texture discrimination. We would like to consider extensions of the spatial-frequency hypothesis that might account for preattentive texture discrimination. We seek to preserve one key feature of the spatial-frequency model: that preattentive texture discrimination makes use only of the distribution of activity levels for each receptive field type. Thus, in order to account for texture discrimination, we must consider a set of receptive fields R^* which contain non-sinusoidal profiles, or, we must postulate that moments of the distribution of the responses higher than order 2 are available to preattentive processing.

More formally, we consider the 'iso-(R^*,p)' hypothesis. This hypothesis states that for some set of receptive fields R^* and some moment p of the distribution of their responses, textures which are iso-(R^*,p) pairs are always unable to support preattentive texture discrimination.

In this terminology, the spatial-frequency hypothesis is equivalent to the iso-$(R^{\mathrm{Fou}},2)$ hypothesis. We will see below that iso-$(R^{\mathrm{Fou}},2)$ texture pairs are automatically iso-$(R^{\mathrm{Fou}},\infty)$ texture pairs as well. Thus, in order to account for texture discrimination on the basis of the outputs of linear receptive fields, we must consider receptive field sets R^* which contain non-sinusoidal profiles. In general, for a set of receptive fields R^* which contains at least some non-sinusoidal profiles, we consider the hypothesis that iso-(R^*,p) texture pairs cannot be discriminated by preattentive vision.

In order for the iso-(R^*,p) hypothesis to be tested, it must be possible to construct iso-(R^*,p) texture pairs for large collections of receptive field types R^* and any finite moment order p. One main result is this construction. For some receptive field sets, we are able to construct iso-(R^*,∞) texture pairs.

Two considerations suggest that the richness of the receptive field set, rather than the number of moments available, is the more critical factor in understanding texture discrimination. Firstly, as we will see below, iso-$(R^{\mathrm{Fou}},2)$ texture pairs are automatically iso-$(R^{\mathrm{Fou}},\infty)$ pairs. Therefore, access to higher moments only matters if some of the receptive field profiles are not sinusoidal. Secondly, an impoverished set of non-sinusoidal receptive fields (uniform Gaussians) fails to explain texture discrimination even when *all* moments of the distribution of responses are available for processing (see Fig. 1 below).

In this model framework, we postulate that preattentive texture discrimination makes use only of the distribution of activity levels for each receptive field type, but not the spatial arrangement of this activity. Thus, textures which are distinct though not discriminable by the preattentive process may be distinguished by later stages of processing which analyze the spatial *arrangement*, as well as the overall distribution, of activity levels in each receptive field population.

The model framework readily allows for incorporation of processing of 'local features'. This corresponds to inclusion of localized receptive field profiles into the set R^*. The model framework also encompasses 'local Fourier analysis', in that such local analysis can be thought of as corresponding to receptive fields whose profiles are a few lobes of a grating, such as Gabor function (Daugman, 1985).

Although the receptive fields of the set R^* are linear, an account of texture discrimination in this format is not necessarily a linear theory. This is because discrimination is hypothesized to occur on the basis of the *distribution* of responses of each receptive field type, rather than merely the average response level. So, for example, nth-order correlations could be used in the discrimination process if one receptive field in R^* consisted of n disks, and the nth moment of the distribution were assumed to be available for discrimination. The main point is that we have attempted to model texture analysis as a *linear* stage in which spatial structure (the shapes of the receptive field profiles) is crucial, followed by a *nonlinear* stage of analysis of response distributions.

The characteristic function

The main tool in the construction of iso-(R^*,∞) and iso-(R^*,p) texture is the characteristic function of the distribution of a receptive field's response to a texture. Here we define the characteristic function and related objects.

A texture A is a set of luminances $A(\mathbf{x})$ for all points on the plane. Since textures are to be considered as *samples* of a stochastic process, we consider them to be samples of an ensemble of textures Ω of similar statistics. We use $\langle\rangle_{\Omega,\mathbf{x}}$ to denote an average over points $\mathbf{x}$ of the plane and over all examples of textures in the ensemble Ω.

We demand three technical properties of a texture ensemble: (i) Correlations are limited in spatial extent. That is, given any $\varepsilon > 0$, there exists a distance D_ε such that spatial correlations (autocorrelations and higher-order analogs) over distances greater than D_ε have absolute value less than ε. (ii) Correlation structure averaged over the ensemble is unchanged after translation in the plane. That is, the texture is (statistically) homogeneous. (iii) For any texture A drawn from the ensemble Ω, averages over $\mathbf{x}$ alone are equivalent to an average over both $\mathbf{x}$ and Ω. These conditions admit a wide variety of textures and exclude various 'pathological' textures for which our analysis would not hold. However, we do not imply that these conditions the minimal ones for our analysis.

A receptive field will be denoted by R, and a particular receptive field within a parametrized set will be denoted R_α. A receptive field is characterized by its sensitivity profile; the value of the sensitivity profile of a receptive field R_α at a position $\mathbf{y}$ relative to a coordinate system fixed with respect to the receptive field will be denoted $R_\alpha(\mathbf{y})$.

Consider placement of the receptive field R at a position $\mathbf{x}$ on the texture A. For each such placement position $\mathbf{x}$, the strength of the response is the convolution of the texture with the receptive field, $\int A(\mathbf{x}+\mathbf{y})R(\mathbf{y})d\mathbf{y}$. We are primarily interested in how these signals are distributed. The main tool for the analysis of this distribution is the characteristic function, which we denote

$$\chi(A; R; z) = \log\langle e^{z\int A(\mathbf{x}+\mathbf{y})R(\mathbf{y})d\mathbf{y}}\rangle_{\Omega,\mathbf{x}}. \quad (1)$$

Properties of the characteristic function. The usual properties associated with characteristic functions will allow us to turn the problem of construction of iso-(R^*,∞) and iso-(R^*,p) textures into a linear problem.

Let the texture $A+B$ denote a texture which is the superposition of statistically-independent textures A and B:

$$(A+B)(\mathbf{x}) = A(\mathbf{x}) + B(\mathbf{x}). \quad (2)$$

Then, the characteristic function of the superposition texture $A+B$ is the sum of the characteristic functions of the component textures:

$$\chi(A+B;R;z) = \chi(A;R;z) + \chi(B;R;z). \quad (3)$$

This is a standard calculation which follows from the convolution formula for Laplace transforms. The hypothesis of statistical independence of the component textures is critical.

Similarly, characteristic functions respect both scalar multiplication of the texture and scalar multiplication of the receptive field. Let ρA denote a texture whose intensity at a point $\mathbf{x}$ is given by

$$(\rho A)(\mathbf{x}) = A(\mathbf{x})\rho. \tag{4}$$

Then,

$$\chi(\rho A;R;z) = \chi(A;R;\rho z). \tag{5}$$

There is a corresponding relationship between the analyzing receptive field R and the generating function variable z. Scalar multiplication of receptive fields is strictly analogous to scalar multiplication of textures (4): ρR is the receptive field whose profile at a point $\mathbf{x}$ is given by $(\rho R)(\mathbf{x}) = R(\mathbf{x})\rho$. Then,

$$\chi(A;\rho R;z) = \chi(A;R;\rho z). \tag{6}$$

Conditions for iso-(R^*, ∞*) and iso-(*R^*, P*) texture pairs.* The characteristic function describes the entire distribution of activities that a texture A elicits among randomly-placed receptive fields R. This is because the distribution of activity can be recovered from the characteristic function by formal inversion of the transform (1), which is essentially a Laplace transform. Thus, for two textures A and B to be iso-(R^*,∞) pairs, they must have identical characteristic functions for all receptive fields in R^*. That is,

$$\chi(A;R_\beta;z) = \chi(B;R_\beta;z) \tag{7}$$

for all receptive fields R_β in R^*.

The weaker condition that two textures are iso-(R^*,p) pairs requires only equality of the first p moments of the distribution of activities for each receptive field. Equality of the first p moments of the activity distribution is equivalent to equality of its first p semi-invariants. One convenient property of the characteristic function is that its derivatives evaluated at the origin correspond to the semi-invariants of the activity distribution. For example,

$$\chi(A;R;z)|_{z=0} = 0, \tag{8a}$$

$$\frac{\partial}{\partial z}\chi(A;R;z)\bigg|_{z=0} = \left\langle \int A(\mathbf{x}+\mathbf{y})R(\mathbf{y})\mathrm{d}\mathbf{y} \right\rangle_{\Omega,\mathbf{x}} = \left\langle A(\mathbf{x}) \right\rangle_{\Omega,\mathbf{x}} \int R(\mathbf{y})\mathrm{d}\mathbf{y}, \tag{8b}$$

and

$$\frac{\partial^2}{\partial z^2}\chi(A;R;z)\bigg|_{z=0} = \left\langle \left(\int A(\mathbf{x}+\mathbf{y})R(\mathbf{y})\mathrm{d}\mathbf{y} \right)^2 \right\rangle_{\Omega,\mathbf{x}} - \left(\left\langle \int A(\mathbf{x}+\mathbf{y})R(\mathbf{y})\mathrm{d}\mathbf{y} \right\rangle_{\Omega\mathbf{x}} \right)^2. \tag{8c}$$

Thus, for texture A and B to be iso-(R^*,p) texture pairs,

$$\frac{\partial^j}{\partial z^j}\chi(A;R_\beta;z)\bigg|_{z=0} = \frac{\partial^j}{\partial z^j}\chi(B;R_\beta;z)\bigg|_{z=0} \tag{9}$$

for all receptive fields R_β in R^* and all orders of derivatives j from 1 to p. Equality of the zeroth derivative at the origin is guaranteed by the fact that all characteristic functions satisfy $\chi(A;R;0) = 0$.

In sum, if two textures are iso-(R^*, ∞) pairs, their characteristic functions must agree. If two textures are iso-(R^*,p) pairs, the first p derivatives (evaluated at the origin) of their characteristic functions must agree.

Poisson textures

Poisson textures, to be defined below, are the spatial analog of Poisson shot noise. Poisson textures form useful building blocks for construction of iso-(R^*,p) texture pairs. We will see that the characteristic function of a Poisson texture has a simple form and exhibits a reciprocity between the profiles which generate the texture and the receptive fields themselves.

A Poisson texture of density λ and profile shape S, to be denoted $A = \text{Pois}[\lambda, S]$, is the superposition of profiles S placed at random in the plane. The position of each profile is specified by the position of an identified point s_0 within the profile S. We require that the positions of these identified points are independent, and the probability that an infinitesimal region dx of the texture contains an identified point of a profile is equal to λdx. Since a profile may be infinite in extent (e.g., a Gaussian), an infinite number of them may be superimposed at a given point. Examples of Poisson textures are shown in Figs 1A and 2A.

The characteristic function of a Poisson texture. We now compute the characteristic function $\chi(\text{Pois}[\lambda,S];R;z)$ of a Poisson texture. For technical reasons, we first consider profiles S that are compact. However, the result makes sense for profiles whose support is noncompact (e.g., Gaussians), and a technical argument shows that the formula derived for compact profiles is typically valid for noncompact profiles as well.

To compute $\chi(\text{Pois}[\lambda,S];R;z)$, we break up the average in (1) into a sum of terms. The nth term represents the contribution from instances in which the receptive field R is overlapped by exactly n copies of the profile S:

$$\langle e^{z\int A(\mathbf{x}+\mathbf{y})R(\mathbf{y})d\mathbf{y}} \rangle_{\Omega,\mathbf{x}} = \sum_{n=0}^{\infty} p_n c_n, \tag{10}$$

where p_n is the probability that R is overlapped by exactly n copies of the profile S, and c_n is the contribution corresponding to this instance. To compute the probability of exactly n overlaps, p_n, we construct a set $K = K(S,R)$ such that a profile S overlaps the receptive field R when (and only when) the identified point s_0 of S is contained in K. K is the support of a function which is the convolution of a function ϕ_S which is unity on S and zero elsewhere, and a function ϕ_R which is unity on R and zero elsewhere. The set K thus depends on the geometry of both S and R.

The probability p_n that the receptive field R intersects with exactly n copies of S is the probability that K contains exactly n identified points of a spatial Poisson process of density λ. This is defined by elementary Poisson statistics, with Poisson parameter equal to the average number of identified points in K. That is,

$$p_n = \frac{(\lambda k)^n}{n!} e^{-\lambda k}, \tag{11}$$

where k is the area of the region $K = K(S,R)$.

The contribution to the sum (10) corresponding to the n-overlap case is given by

$$c_n = \langle e^{z\int\sum_{j=1}^{n} S(\mathbf{x}_j+\mathbf{y})R(\mathbf{y})d\mathbf{y}} \rangle_{\mathbf{x}_j \varepsilon K}. \tag{12}$$

(For $n = 0$, overlap never occurs, and $c_0 = 1$.) Each term in the sum on the right-hand-side of (12) is independent. Therefore, the right-hand-side of (12) factors into the product of n identical terms. Each term, which is an average over the set K, can be replaced by an integral:

$$c_n = \prod_{j=1}^{n} \langle e^{z\int S(x_j+y)R(y)dy} \rangle_{x_j \varepsilon K} = \left(\frac{1}{k} \int_K e^{z\int S(x+y)R(y)dy} dx \right)^n. \tag{13}$$

We now substitute the equation (11) for the Poisson probability of n overlaps p_n and (13) for the contribution c_n into (10):

$$\langle e^{z\int A(x+y)R(y)dy} \rangle_{\Omega,x} = \sum_{n=0}^{\infty} \frac{(\lambda k)^n}{n!} e^{-\lambda k} \left(\frac{1}{k} \int_K e^{z\int S(x+y)R(y)dy} dx \right)^n = e^{-\lambda k} e^{(\lambda \int_K e^{z\int S(x+y)R(y)dy} dx)}. \tag{14}$$

The characteristic function (1) is the logarithm of the quantity (14):

$$\chi(\mathrm{Pois}[\lambda, S]; R; z) = \lambda \left(-k + \int_K e^{z\int S(x+y)R(y)dy} dx \right) = \lambda \int_K (e^{z\int S(x+y)R(y)dy} - 1) dx. \tag{15}$$

The integral over the area of overlap K may be replaced by an integral over the entire plane, because the integrand is zero outside of the overlap region. The result (15) was derived only for receptive fields of compact support. However, receptive fields of infinite support can be considered as the limiting case of an expanding series of receptive fields with compact support. (The technical proviso is that convolution of a receptive field profile R and the basic element S of the Poisson texture is integrable.) Thus, for all such 'reasonable' receptive field profiles,

$$\chi(\mathrm{Pois}[\lambda, S]; R; z) = \lambda \int (e^{z\int S(x+y)R(y)dy} - 1) dx. \tag{16}$$

A simple relationship for the characteristic function of Poisson textures that differ either in the frequency λ or the scale-factor σ follows from (16):

$$\chi(\mathrm{Pois}[\lambda, \sigma S]; R; z) = \lambda \chi(\mathrm{Pois}[1, S]; R; \sigma z). \tag{17}$$

In both (16) and (17), the density λ must be non-negative.

The semi-invariants (8) of the response distribution are equal to the derivatives of the characteristic function (16) at the origin:

$$\frac{\partial^j}{\partial z^j} \chi(\mathrm{Pois}[\lambda, S]; R; z) \bigg|_{z=0} = \lambda \int \left(\int S(x+y)R(y)dy \right)^j dx. \tag{18}$$

The first two derivatives (mean and variance of the response distribution (8) are given by:

$$\frac{\partial}{\partial z} \chi(\mathrm{Pois}[\lambda, S]; R; z) \bigg|_{z=0} = \left(\int S(x) dx \right) \left(\int R(y) dy \right) \tag{19}$$

and

$$\frac{\partial^2}{\partial z^2}\chi(\mathrm{Pois}[\lambda,S];R;z)\bigg|_{z=0} = \int\left(\int S(\mathrm{x}+\mathrm{y})R(\mathrm{y})\,\mathrm{dy}\right)^2\mathrm{dx}. \tag{20}$$

Construction of iso-(R^*,∞) texture pairs from Poisson textures

Let us say we are given a texture A and a set of receptive field profiles R^* with members R_α. It is possible to create distinct texture $\bar{A}$, such that A and $\bar{A}$ form an iso-(R^*,∞) texture pair? The larger the set of profiles R^*, the greater the constraints imposed on $\bar{A}$. This suggests that as R^* grows, we must have more and more degrees of freedom to construct $\bar{A}$. For this reason, we will use Poisson textures based on the elements of R^* itself to provide candidates for $\bar{A}$.

We now construct a candidate for $\bar{A}$. Consider a Poisson texture $\mathrm{Pois}[\lambda,\rho R_\alpha]$. Recall that ρR_α is a receptive field R_α in R^* whose intensity profile (contrast) has been scaled by the quantity ρ. $\mathrm{Pois}[\lambda,\rho R_\alpha]$ is composed of many copies of the scaled profile ρR_α, randomly placed on the plane with a density λ. We construct $\bar{A}$ as a *superposition* of such textures, with one texture in the superposition for each choice α of a profile in R^* and each intensity-scaling ρ. We allow the density λ of each texture in the superposition to depend both on α and ρ. This superposition of textures is a limit of texture sums, where texture sums are interpreted according to the definition (2). This superposition of textures may therefore be represented as an integral

$$\bar{A} = \iint \mathrm{Pois}[\lambda(\alpha,\rho),\rho R_\alpha]\,\mathrm{d}\alpha\,\mathrm{d}\rho. \tag{21}$$

For receptive field sets R^* which are discretely-parametrized, the integral over α in (21) becomes a sum.

The superposition (21) provides a number of degrees of freedom appropriate for the set of receptive fields R^*. We now seek a set of non-negative densities $\lambda(\alpha,\rho)$ such that $\bar{A}$ and A are iso-(R^*,∞) texture pairs.

The properties ((3) and (5)) of the characteristic function and the formulae (16) and (17) provide for computation of the characteristic function of the texture $\bar{A}$ as specified by (21):

$$\begin{aligned}\chi(\bar{A};R_\beta;z) &= \iint \chi(\mathrm{Pois}[\lambda(\alpha,\rho),\rho R_\alpha];R_\beta;z)\,\mathrm{d}\alpha\,\mathrm{d}\rho\\ &= \iint \lambda(\alpha,\rho)\left(\int(\mathrm{e}^{\rho z\int R_\alpha(\mathrm{x}+\mathrm{y})R_\beta(\mathrm{y})\mathrm{dy}}-1)\,\mathrm{dx}\right)\mathrm{d}\alpha\,\mathrm{d}\rho,\end{aligned} \tag{22}$$

where the integral over α covers the parameter space of R^* and the integral over ρ covers all intensity-scalings.

The requirement that A and $\bar{A}$ are iso-(R^*,∞) texture pairs is that their characteristic functions agree (Equation (7)). The characteristic function of A is assumed known; the characteristic function of $\bar{A}$ is the weighted sum of the characteristic functions of its component Poisson textures (22). The iso-(R^*,∞) condition is:

$$\chi(A;R_\beta;z) = \iint \lambda(\alpha,\rho)\left(\int(\mathrm{e}^{\rho z\int R_\alpha(\mathrm{x}+\mathrm{y})R_\beta(\mathrm{y})\mathrm{dy}}-1)\,\mathrm{dx}\right)\mathrm{d}\alpha\,\mathrm{d}\rho. \tag{23}$$

for all profiles R_β in R^*. The equation (23), considered as an equation for $\lambda(\alpha,\rho)$, is a linear integral equation. The dependence on the variable pair (ρ,z) is essentially that of a convolution (after log-transformation). The dependence on the variable pair (α,β) is also symmetric but typically more complex than a convolution; its details depend on the receptive field set R^*. Furthermore, provided that the set of receptive fields R^* is held fixed, the descriptor $\lambda(\alpha,\rho)$ of a solution-texture $\bar{A}$ is linear in the texture A.

A solution $\lambda(\alpha,\rho)$ to the fundamental Equation (23) specifies a texture $\bar{A}$ according to (21), provided that the values $\lambda(\alpha,\rho)$ are non-negative. If the solution to (23) includes negative values for $\lambda(\alpha,\rho)$, technical difficulties arise because the formula (16) applies only for *non-negative* values of the density λ. Negation (contrast-inversion) of a texture does not result in negation of the characteristic function.

These difficulties may be overcome as follows. Any solution $\lambda(\alpha,\rho)$ may be interpreted as a difference of two non-negative components,

$$\lambda(\alpha,\rho) = \lambda_+(\alpha,\rho) - \lambda_-(\alpha,\rho). \tag{24}$$

Through the construction (21), $\lambda_+(\alpha,\rho)$ specifies a texture $\bar{A}_+$, and $\lambda_-(\alpha,\rho)$ specifies a texture $\bar{A}_-$. Consider the composite texture $A+\bar{A}_-$. Since $\bar{A}_-$ is a superposition of Poisson textures, it is statistically independent of the texture A. According to (3), the characteristic function of the texture $A+\bar{A}_-$ is the sum of the characteristic function of A and that of $\bar{A}$. Because of the decomposition (24), $\lambda_+(\alpha,\rho)$ solves (23) for the composite texture $A+\bar{A}_-$. That is, $\bar{A}_+$ and $A+\bar{A}_-$ are iso-(R^*,∞) texture pairs. Thus, any solution of (23) provides a nontrivial iso-(R^*,∞) texture pair, either directly (via (21)), or via the above construction.

An iso-(R^*,∞*) texture pair.* We next consider a special case of the fundamental Equation (23) to demonstrate that there are indeed nontrivial examples of iso-(R^*,∞) textures. The set R^* of receptive fields will consist of Gaussian profiles. We let the receptive field R_α denote a 2-dimensional Gaussian with variance-covariance matrix V_α. This is, $R_\alpha(\mathbf{x}) = \mathrm{Gau}(V_\alpha,\mathbf{x})$, where

$$\mathrm{Gau}(V,\mathbf{x}) = \frac{1}{2\pi\det^{1/2}V}\mathrm{e}^{-V^{-1}\mathbf{x}\cdot\mathbf{x}/2}. \tag{25}$$

Simplifications of (23) follow from the well-known fact that the convolution of two Gaussians is again a Gaussian, whose variance-covariance matrix is the sum of those of the convolved Gaussians:

$$\begin{aligned}\int R_\alpha(\mathbf{x}+\mathbf{y})R_\beta(\mathbf{y})\,\mathrm{d}\mathbf{y} &= \int \mathrm{Gau}(V_\alpha,\mathbf{x}+\mathbf{y})\mathrm{Gau}(V_\beta,\mathbf{y})\,\mathrm{d}\mathbf{y}\\ &= \mathrm{Gau}(V_\alpha+V_\beta,\mathbf{x}).\end{aligned} \tag{26}$$

For a set of Gaussian receptive fields R^*, the right hand side of the fundamental Equation (23) becomes a convolution in the (α,β) variables, as well as in the (ρ,z) variables. To see this, we calculate:

$$\begin{aligned}\chi(\mathrm{Pois}[\lambda,\rho R_\alpha];R_\beta;z] &= \lambda\,\chi(\mathrm{Pois}[1,R_\alpha];R_\beta;\rho z)\\ &= \lambda\int(\mathrm{e}^{\rho z\,\mathrm{Gau}(V_\alpha+V_\beta,\mathbf{x})}-1)\,\mathrm{d}\mathbf{x}\\ &= \lambda\det{}^{1/2}(V_\alpha+V_\beta)\int(\mathrm{e}^{(\rho z/2\pi\det^{1/2}(V_\alpha+V_\beta))\mathrm{e}^{-|\mathbf{u}|^2/2}}-1)\,\mathrm{d}\mathbf{u}\\ &= \lambda\det{}^{1/2}(V_\alpha+V_\beta)F\left(\frac{\rho z}{\det^{1/2}(V_\alpha+V_\beta)}\right).\end{aligned} \tag{27}$$

The first equality is a consequence of (17). The second equality follows from application of the convolution formula (26) to the general formula (16) for the characteristic function of a Poisson texture. The third equality follows from a change of variables $\mathbf{x} = M^{-1}\mathbf{u}$ where M is a self-adjoint matrix such that $(V_\alpha + V_\beta)^{-1} = M^2$. This transforms the Gaussian $\mathrm{Gau}(V_\alpha + V_\beta, \mathbf{x})$ into the standard Gaussian (of unit variance), and transforms the volume element $\mathrm{d}\mathbf{x}$ to $\det^{-1} M \cdot \mathrm{d}\mathbf{u}$ with $\det^{-1} M = \det^{1/2}(V_\alpha + V_\beta)$. Thus, all characteristic functions may be calculated in terms of a common function

$$F(z) = \int (\mathrm{e}^{(z/2\pi)\mathrm{e}^{-|\mathbf{u}|^2/2}} - 1)\,\mathrm{d}\mathbf{u}. \tag{28}$$

For two-dimensional textures, the common function (28) may be transformed via polar coordinates to the exponential integral function:

$$\begin{aligned} F(z) &= 2\pi \int_0^\infty r(\mathrm{e}^{(z/2\pi)\mathrm{e}^{-r^2/2}} - 1)\,\mathrm{d}r \\ &= 2\pi \int_0^{z/2\pi} \frac{\mathrm{e}^t - 1}{t}\,\mathrm{d}t. \end{aligned} \tag{29}$$

Now let us assume that the texture A (for which we seek an iso-(R^*, ∞) texture pair) consists of a Poisson texture based on a Gaussian profile R_γ *not* contained in R^*: $A = \mathrm{Pois}[\mu, \sigma R_\gamma]$. In this case, the characteristic function of A (the left-hand-side of Equation (23)) has a form similar to that of the right-hand-side. We reuse the result (27) to find

$$\begin{aligned} \chi(A; R_\beta; z) &= \chi(\mathrm{Pois}[\mu, \sigma R_\gamma]; R_\beta; z) \\ &= \mu \det{}^{1/2}(V_\gamma + V_\beta) F\left(\frac{\sigma z}{\det^{1/2}(V_\gamma + V_\beta)}\right). \end{aligned} \tag{30}$$

The fundamental Equation (23) for iso-(R^*, ∞) textures becomes:

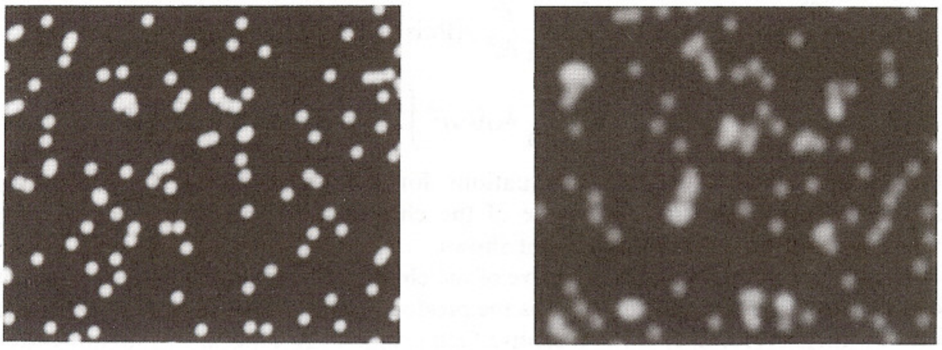
$$\begin{aligned} &\mu \det{}^{1/2}(V_\gamma + V_\beta) F\left(\frac{\sigma z}{\det^{1/2}(V_\gamma + V_\beta)}\right) \\ &= \iint \lambda(\alpha, \rho) \det{}^{1/2}(V_\alpha + V_\beta) F\left(\frac{\rho z}{\det^{1/2}(V_\alpha + V_\beta)}\right) \mathrm{d}\alpha\,\mathrm{d}\rho. \end{aligned} \tag{31}$$

The simplest instance of this equation is a set of textures R^* which contains just one Gaussian profile R_β. In this case, there is no integral over α in (31) ($\alpha = \beta$), and $\lambda(\alpha, \rho) = \lambda(\rho)$. One may verify a solution of (31) by direct substitution:

$$\lambda(\rho) = \mu \frac{\det^{1/2}(V_\gamma + V_\beta)}{\det^{1/2}(2V_\beta)} \delta\left(\rho - \sigma \frac{\det^{1/2}(2V_\beta)}{\det^{1/2}(V_\gamma + V_\beta)}\right). \tag{32}$$

This is a very special case: an iso-(R^*, ∞) relation between two Poisson textures, constructed from distinct Gaussian profiles. Nevertheless, it shows that there are nontrivial examples of iso-(R^*, ∞) texture pairs. An example of this texture pair for $V_\beta = 4V_\gamma$ is shown in Fig. 1.

The solution (32) states that a texture composed of Gaussians of variance V_γ may be matched by a texture composed of Gaussians of a distinct variance V_β. For circular Gaussians, if the spread (V_γ) of the Gaussians in the original texture A is smaller than



A B

Figure 1. An iso-(R^*, ∞) texture pair, for a receptive field set R^* which contains a single Gaussian profile of variance V_β. Part A is a Poisson texture constructed from a Gaussian profile whose variance is V_γ. Part B is a Poisson texture constructed from a Gaussian profile whose variance is $V_\beta = 4V_\gamma$, according to (21) and (32).

the spread (V_β) of the Gaussians in the matching texture $\bar{A}$, then the matching texture will be composed of a smaller number of profiles per unit area, but with each profile at a higher contrast-scaling (volume). The larger spatial extent of the profiles in the matching texture more than compensates for their larger volume, and thus, the peak intensity of the profiles in $\bar{A}$ is smaller than the peak intensity of the profiles in A (Fig. 1). The number-density and contrast-scalings vary inversely, so that both textures have the same mean luminances. The power spectra of the two textures are not guaranteed to be equal. This is because the receptive-field set R^* does not include sinusoidal profiles.

*Construction of iso-(*R*,p*) texture pairs*

For the special case of a single Gaussian (above), the integral Equation (23) was solved by inspection. For a more general receptive field set R^*, it may be difficult to solve the integral Equation (23) in closed form. Indeed, because the equation is similar to a convolution, there is no guarantee that useful solutions will exist. However, these technical difficulties can be circumvented if we relax the iso-(R^*, ∞) constraint to an iso-(R^*, p) constraint.

To highlight the difference between the iso-(R^*, p) construction and the iso-(R^*, ∞) construction, we assume that the receptive field set R^* is discretely parametrized, and contains receptive fields R_α ($\alpha = 1, 2, \ldots, N$). We again consider a superposition of Poisson textures. This superposition covers a discrete set of Np intensity-scalings $\rho_{\alpha j}$ ($\alpha = 1, 2, \ldots, N$; $j = 1, \ldots, p$), rather than the continuous superposition (21) used for the iso-(R^*, ∞) construction. The candidate texture $\bar{A}$ is now a finite superposition of textures, rather than the continuous superposition (21):

$$\bar{A} = \sum_{\alpha=1}^{N} \sum_{j=1}^{p} \mathrm{Pois}\left[\lambda_{\alpha j}, \rho_{\alpha j} R_\alpha\right]. \tag{33}$$

The condition that the textures A and $\bar{A}$ are iso-(R^*, p) textures is (9) that the first p derivatives of the characteristic functions agree. That is (using (18) to evaluate the derivatives of the characteristic function of a Poisson texture),

$$\left.\frac{\partial^k}{\partial z^k}\chi(A;R_\beta;z)\right|_{z=0} = \sum_{\alpha=1}^{N}\sum_{j=1}^{p}\left.\frac{\partial^k}{\partial z^k}\chi(\mathrm{Pois}[\lambda_{\alpha j},\rho_{\alpha j}R_\alpha];R_\beta;z)\right|_{z=0}$$
$$= \sum_{\alpha=1}^{N}\sum_{j=1}^{p}\lambda_{\alpha j}(\rho_{\alpha j})^k\int\left(\int R_\alpha(\mathbf{x}+\mathbf{y})R_\beta(\mathbf{y})\,d\mathbf{y}\right)^k d\mathbf{x}. \qquad (34)$$

This is a system of Np linear equations for the densities $\lambda_{\alpha j}$. The equations corresponding to the first derivative of the characteristic function ($k=1$) can be eliminated, as the following argument shows.

According to (8b), the first derivative of the characteristic function of the texture A with respective to a receptive field R is the product of the mean value of the texture A and the integral (volume) of the receptive field profile $R(\mathbf{x})$. Thus, if two textures have the same means, then the first derivatives of their characteristic functions with respect to any receptive field must agree. Now, assume that a solution to all equations of (34) $2 \leqslant k \leqslant p$ has been found. The only way that the corresponding texture $\bar{A}_0$ may fail to be an iso-(R^*,p) texture pair is that the first derivative of its characteristic functions may not match those of the texture A. But, this mismatch may be corrected by adding a uniform quantity to $\bar{A}_0$, so that its mean value agrees with that of the original texture A. Since the higher derivatives of the characteristic function only depend on the *central* moments of the response distribution, the values of these higher derivatives are not changed by the addition of a constant value to $\bar{A}_0$.

While there is no guarantee that the equations for $2 \leqslant k \leqslant p$ of (34) are nonsingular, there is freedom to choose the values of the scale factors $\rho_{\alpha j}$. Thus, the generic situation is that the system is *underdetermined*, because of the freedom to choose these scale factors.

We consider the $p=2$ case in more detail, and take $\rho_{\alpha 1}=0$, $\rho_{\alpha 2}=\rho_\alpha$, and $\lambda_{\alpha 2}=\lambda_\alpha$. The above argument indicates that the equations (34) for $k=1$ may be neglected. For $k=2$, the equations (34) reduce to

$$\left.\frac{\partial^2}{\partial z^2}\chi(A;R_\beta;z)\right|_{z=0} = \sum_{\alpha=1}^{N}\lambda_\alpha\rho_\alpha^2\int\left(\int R_\alpha(\mathbf{x}+\mathbf{y})R_\beta(\mathbf{y})\,d\mathbf{y}\right)^2 d\mathbf{x}. \qquad (35)$$

This set of equations is a linear system of equations for the N quantities $q_\alpha=\lambda_\alpha\rho_\alpha^2$. Thus, provided that (35) is a nonsingular system, there are N degrees of freedom (corresponding to the N arbitrary choices for ρ_α^2) for the texture $\bar{A}$.

Examples of construction of iso-(R^*,2) and iso-(R^*,3) textures are given in the Appendix and Fig. 2.

Sinusoidal profiles: redundancy of higher moments

We began with the observation that the power spectrum of a texture consists of the second moments of the distributions of activities of randomly-placed sinusoidal receptive fields. Since these statistics do not suffice to account for preattentive texture discrimination, we considered generalizations in which higher moments replaced the second moment, or in which other receptive fields replaced the sinusoidal profiles. Here we demonstrate that higher moments do not provide any additional information if the receptive fields are all sinusoidal and of indefinitely large spatial extent.

We consider a receptive field profile which corresponds to a spatial frequency $\mathbf{f}$. (Spatial frequency may be represented by a vector, whose magnitude $f=|\mathbf{f}|$ is the (scalar) spatial frequency, and whose direction is normal to the bars of the corresponding sine grating.) As before, we consider receptive fields whose spatial extent, L, is large in comparison with the spatial period $1/f$, and assume that the spatial weighting is zero at distances greater than L. That is, the spatial weighting corresponding to a profile R_f may be written

$$R_f(\mathbf{y}) = \begin{cases} \dfrac{1}{\pi^{1/2}L}\cos(2\pi\mathbf{y}\cdot\mathbf{f}), & |\mathbf{y}| < L \\ 0, & |\mathbf{y}| > L. \end{cases} \tag{36}$$

We assume that the spatial extent L of the receptive field is large in comparison to both the spatial period $1/f$ and the correlation length of the texture A under consideration.

Let the value of the spatial frequency spectrum of the texture A at the spatial frequency $\mathbf{f}$ be denoted by $\tilde{A}(\mathbf{f})$. That is,

$$\lim_{L\to\infty}\left\langle\left(\int A(\mathbf{x}+\mathbf{y})R_{\mathbf{f}}(\mathbf{y})\,d\mathbf{y}\right)^2\right\rangle_{\Omega,\mathbf{x}} = \tilde{A}(\mathbf{f}). \tag{37}$$

To estimate the distribution of the quantity $\int A(\mathbf{x}+\mathbf{y})R_{\mathbf{f}}(\mathbf{y})\,d\mathbf{y}$, we subdivide the support of the receptive field $R_{\mathbf{f}}$ into multiple disjoint subregions $K_1, K_2, \ldots, K_M$:

$$\int A(\mathbf{x}+\mathbf{y})R_{\mathbf{f}}(\mathbf{y})\,d\mathbf{y} = \sum_{j=1}^{M}\int_{K_j} A(\mathbf{x}+\mathbf{y})R_{\mathbf{f}}(\mathbf{y})\,d\mathbf{y}. \tag{38}$$

The subregions K_j may be taken to be the intersection of the disk $|\mathbf{y}| < L$ with a rectangular lattice (independent of L). Then, although the terms of (38) corresponding to neighboring cells of the lattice may be correlated, terms which correspond to distant cells are uncorrelated.

In the limit that the size L of the receptive field grows without bound, more and more of the terms of (38) are uncorrelated with each other. Furthermore, all of these terms have equal variances, since the texture A is statistically homogeneous. Thus, the central limit theorem may be invoked. Therefore (for large L), the quantity $\int A(\mathbf{x}+\mathbf{y})R_{\mathbf{f}}(\mathbf{y})\,d\mathbf{y}$ is distributed as a Gaussian random variable. The variance of this distribution is given by (37). The mean of this distribution must be zero, since replacement of the receptive field by its negative (which would negate the mean of the distribution) is equivalent to displacement of the receptive field by half a spatial wavelength (which must leave the mean unchanged, because of the assumption of homogeneity). Thus, the distribution of the quantity $h = \int A(\mathbf{x}+\mathbf{y})R_{\mathbf{f}}(\mathbf{y})\,d\mathbf{y}$ is a Gaussian of mean zero and variance $\tilde{A}(f)$, in the limit of a large receptive field. The corresponding characteristic function is therefore given by

$$\begin{aligned}\chi(A; R_{\mathbf{f}}; z) &= \log\left\langle e^{z\int A(\mathbf{x}+\mathbf{y})R_{\mathbf{f}}(\mathbf{y})d\mathbf{y}}\right\rangle_{\Omega,\mathbf{x}} \\ &= \log\int e^{zh}\frac{1}{(2\pi\tilde{A}(\mathbf{f}))^{1/2}}e^{-h^2/2\tilde{A}(\mathbf{f})}\,dh \\ &= \tilde{A}(\mathbf{f})z^2/2.\end{aligned} \tag{39}$$

Thus, through an application of the central limit theorem, we have shown that for a receptive field whose profile approaches an extensive sinusoid, the distribution of responses to a texture is a Gaussian whose variance is the value of the power spectrum of that texture at the corresponding spatial frequency. Consequently, the higher moments provide no additional information for such receptive fields.

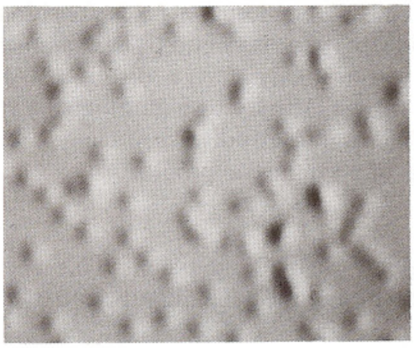
DISCUSSION

Any visual stimulus may be decomposed into its spatial Fourier components. For many visual stimuli (Julesz, 1962; Rentschler and Treutwein, 1985; Rentschler *et al.*, 1988), early processing of visual patterns appears to ignore spatial-phase relationships, and to depend only on the amplitudes of the spatial Fourier components. However, there are many exceptions to the claim that texture discrimination depends solely on spatial-frequency content (Caelli and Julesz, 1978; Caelli *et al.*, 1978; Julesz *et al.*, 1978; Victor and Brodie, 1978; Gagalowitz, 1981; Gagalowitz and Ma, 1985). It is natural to wonder whether the 'exceptions' are all consequences of the fact that an actual visual mechanism which subserves Fourier analysis cannot have infinitesimal bandwidth, or, equivalently, must have only finite spatial extent (Daugman, 1985). One consequence of the necessarily finite size and nonzero bandwidth of actual receptive fields is that finite spatial profiles allow each receptive field to convey spatial-position as well as spatial-frequency information.

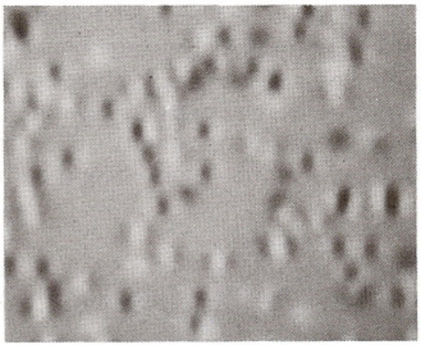
There is a second, distinct consequence of finite receptive field size. For infinite sinusoidal receptive fields, second-order statistics provide complete information on the response distribution; higher moments are redundant. For receptive fields of finite size, higher-order moments of the response distribution are no longer redundant. The techniques outlined above provide an experimental approach to determine whether the information content in the higher moments is utilized by preattentive processes. This information may be dissociated from positional information. As seen in Fig. 1, for a very impoverished receptive-field set (which consists of only one Gaussian profile), iso-(R^*, ∞) textures are highly discriminable. This suggests that processing of high-order moments does not overcome the limitations of a very limited receptive field set. To further investigate this question, a converse approach is required as well: for example, a comparison of the discriminability of iso-$(R^*, 2)$ textures with that of iso-$(R^*, 3)$ textures for receptive-field sets which contain a rich variety of spatial profiles.

Another possible reason that a strict spatial frequency hypothesis might fail is that detection of local features is a critical element. This was the hypothesis put forward by Julesz and coworkers (Caelli and Julesz, 1978; Caelli *et al.* 1978; Bergen and Julesz, 1983). A rudimentary form of this hypothesis is examined by the textures in Fig. 2. Figure 2 shows three iso-$(R^*,3)$ textures with respect to a receptive field set which consists of a Gaussian profile and its second spatial derivative. The shading of all the elements of Fig. 2A is on the same side, while the shading in Fig. 2B is even-symmetric, and the shadeing in Fig. 2C, which is random, is statistically even-symmetric, Figures 2A and B are readily discriminated. The discriminability of Figs 2A and C, although weaker, is remarkable in view of the fact that they are iso-(R^*, ∞) pairs, for the large receptive field set which consists of all gratings and all even-symmetric local profiles. This set includes all even-symmetric Gabor functions and all even-order profiles $D\mathrm{Gau}_\alpha$. This suggests that an account of texture processing requires not merely a diverse set of gratings and local profiles, but also that the local profiles are of varied symmetry.

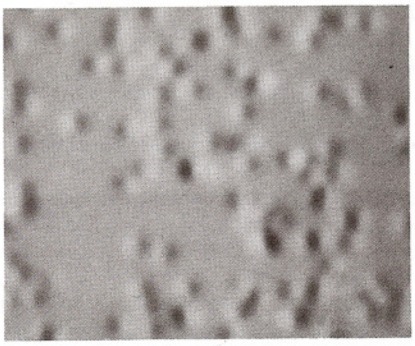
These examples are meant to be heuristic, and certainly do not constitute a rigorous evaluation. In general, the hypothesis that texture discrimination is driven by spatial-frequency analysis in conjunction with local feature detection may be tested by the construction of iso-(R^*,p) textures pairs, for a receptive-field set R^* which includes grating profiles augmented by profiles which match the postulated local features. For



A



B



C

Figure 2. A set of iso-(R^*, 3) textures, for a receptive field set R^* which consists of a Gaussian $D\text{Gau}_0$ and its second (spatial) derivative $D\text{Gau}_2$. Part A is a Poisson texture constructed from a derivative-of-Gaussian profile $D\text{Gau}_1$. Part B is derived from a superposition (33) of $D\text{Gau}_0$ and $D\text{Gau}_2$. The densities and contrasts of these profiles are chosen according to (A5), and the symmetrization (A8) is then performed. Part C consists of profiles $D\text{Gau}_1$ symmetrized according to (A8). Parts A and C are iso-(R^*, ∞) textures for all receptive fields which have even-symmetric profiles.

example, the notion that Gabor-function receptive fields (Daugman, 1985; Rentscher *et al.*, 1988) are critical may be tested by examining discriminability of iso-(R^*,p) texture pairs, for a receptive-field set R^* which consists of Gabor profiles. The model for texture segregation proposed by Sutter *et al.* (1988) also falls into this general framework.

For receptive field sets R^* which contain many elements, numerical solution of (23) or (34) is likely to require extensive computation. We emphasize that the construction of iso-(R^*,p) textures by the superposition of Poisson textures is only one approach to texture synthesis. The construction is important because it demonstrates the existence of such textures (and hence, the ability to test theories); there is no claim that the construction is in any sense the most efficient construction.

Our main points are: (i) theories of texture perception based on spatial-frequency analysis, or on local features, or on a combination of these processes, may be placed in a common framework that may be rigorously tested; and (ii) the two very different consequences of filtering by spatially-limited sinusoidal receptive fields (position information and higher-order moment information) may be dissociated. The examples presented suggest two hypothesis concerning texture analysis: (i) a diverse set of receptive fields appears to be more important than sensitivity to high moments of response distributions; and (ii) a set of receptive fields sufficiently diverse to account for texture perception must contain both even- and odd-symmetric profiles.

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APPENDIX

Construction of iso-(R^*,2) textures for derivatives of Gaussians

We consider a receptive field set R^* which consists of derivatives (of various orders) of a Gaussian of fixed variance V. For simplicity, we only consider circularly-symmetric Gaussians, whose variance/covariance matrix V is a scalar quantity v times the identity matrix I. That is, the profiles R_α are parametrized by an integer index α as follows:

$$R_\alpha(\mathbf{x}) = D\mathrm{Gau}_\alpha(V, \mathbf{x}) = (-1)^\alpha v^{\alpha/2} M(\alpha) \frac{\partial^\alpha}{\partial x_1^\alpha} \mathrm{Gau}(V, \mathbf{x}), \tag{A1}$$

where $M(\alpha)$ is a normalization factor

$$M(\alpha) = \frac{2^{\alpha/2}\Gamma(\alpha/2+1)}{\Gamma(\alpha+1)}. \tag{A2}$$

The profile $D\mathrm{Gau}_0$ is simply a Gaussian. The profile $D\mathrm{Gau}_1$ is a differentiated Gaussian ('edge-detector'). The profile $D\mathrm{Gau}_2$ is the second derivative (in one dimension) of a Gaussian, and is similar to a difference-of-Gaussians or a Laplacian-of-Gaussian profile. In general, the profile $D\mathrm{Gau}_n$ consists of a polynomial with n zero-crossings (a Hermite polynomial) windowed by a Gaussian, and thus resembles a Gabor function. The normalization factor $M(\alpha)$ defined in Equation (A2) ensures that profiles of even order have height $\pm 1/2\pi v$ at the origin, and that profiles of odd order are comparably scaled.

To calculate the coefficients that occur in the right-hand side of (35), we observe that

$$\int D\mathrm{Gau}_\alpha(V, \mathbf{x}+\mathbf{y}) D\mathrm{Gau}_\beta(V, \mathbf{y})\, d\mathbf{y} = (-1)^\beta 2^{-(\alpha+\beta)/2} \frac{M(\alpha)M(\beta)}{M(\alpha+\beta)} D\mathrm{Gau}_{\alpha+\beta}(2V, \mathbf{x}). \tag{A3}$$

This result (which is a generalization of the convolution formula (26)) may be used twice to calculate the integral of the *square* of the convolution of two receptive field profiles, as required for (35):

$$\int \left(\int D\mathrm{Gau}_\alpha(V, \mathbf{x}+\mathbf{y}) D\mathrm{Gau}_\beta(V, \mathbf{y})\, d\mathbf{y} \right)^2 d\mathbf{x} \tag{A4}$$

$$= 2^{-(\alpha+\beta)} \left(\frac{M(\alpha)M(\beta)}{M(\alpha+\beta)} \right)^2 \int \left(D\mathrm{Gau}_{\alpha+\beta}(2V, \mathbf{x}) \right)^2 d\mathbf{x}$$

$$= \frac{1}{8\pi v} 4^{-(\alpha+\beta)} \frac{M(\alpha)^2 M(\beta)^2}{M(2\alpha+2\beta)}.$$

The condition for iso-(R^*, 2) textures now becomes:

$$\left. \frac{\partial^2}{\partial z^2} \chi(A; R_\beta; z) \right|_{z=0} = \sum_{\alpha=1}^{N} \lambda_\alpha \rho_\alpha^2 \cdot \frac{1}{8\pi v} 4^{-(\alpha+\beta)} \frac{M(\alpha)^2 M(\beta)^2}{M(2\alpha+2\beta)}. \tag{A5}$$

This analysis extends directly to profiles which are derivatives of Gaussians in two dimensions, as well as to sets of profiles based on Gaussians of distinct variances. For this receptive field set, we parametrize the receptive fields R_α by a pair of integers $\alpha = (\alpha_1, \alpha_2)$ and define

$$R_\alpha(\mathbf{x}) = D\mathrm{Gau}_\alpha(V, \mathbf{x}) = (-1)^{\alpha_1+\alpha_2} v^{(\alpha_1+\alpha_2)/2} M(\alpha_1) M(\alpha_2) \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \frac{\partial^{\alpha_2}}{\partial x_2^{\alpha_2}} \mathrm{Gau}(V, \mathbf{x}). \tag{A6}$$

The iso-(R^*, 2) condition becomes

$$\left. \frac{\partial^2}{\partial z^2} \chi(A; R_\beta; z) \right|_{z=0} = \sum_\alpha \lambda_\alpha \rho_\alpha^2 \cdot \tag{A7}$$

$$\frac{1}{4\pi(v_\alpha + v_\beta)} \frac{v_\alpha^{\alpha_1+\alpha_2} v_\beta^{\beta_1+\beta_2}}{(2v_\alpha + 2v_\beta)^{\alpha_1+\alpha_2+\beta_1+\beta_2}} \frac{M(\alpha_1)^2 M(\beta_1)^2}{M(2\alpha_1+2\beta_1)} \frac{M(\alpha_2)^2 M(\beta_2)^2}{M(2\alpha_2+2\beta_2)}.$$

A symmetry, and construction of iso-(R, 3) textures*

The equations (A5) or (A7) may be used to construct iso-(R^*, 2) textures with respect to a set of receptive fields which consists of Gaussian profiles and their derivatives. However, there is a simpler way to construct such textures. This is simply a consequence of the fact (8c) that the variance of a response distribution is independent of the signature (polarity) of the texture, or of the receptive field. Thus, provided that the mean of a texture A is zero, the texture A and its contrast-reversal -A are iso-(R^*, 2) textures for any receptive field set R^*.

Another application of this symmetry principle is as follows. Let $\bar{B}$ be any superposition of Poisson textures. Consider a texture $\bar{B}_{1/2}$ which is the superposition of the Poisson textures of $\bar{B}$, but with each profile present at half the density specified by (35). The characteristic function of $\bar{B}_{1/2}$ is half that of $\bar{B}$ (Equation (17)). Consider a second texture $\bar{B}'_{1/2}$, which is a superposition of the Poisson textures of $\bar{B}_{1/2}$ but in which each profile is contrast-reversed. The *even* semi-invariants (variance, kurtosis, etc.) of $\bar{B}_{1/2}$ and $\bar{B}'_{1/2}$ are equal to half those of the texture $\bar{B}$, while the *odd* semi-invariants (mean, skewness, etc.) of $\bar{B}_{1/2}$ and $\bar{B}'_{1/2}$ are equal in magnitude but opposite in sign. Now we superimpose $\bar{B}_{1/2}$ and $\bar{B}'_{1/2}$ to form sym($\bar{B}$);

$$\mathrm{sym}(\bar{B}) = \bar{B}_{1/2} + \bar{B}'_{1/2}. \tag{A8}$$

Since $\bar{B}_{1/2}$ and $\bar{B}'_{1/2}$ are both superpositions of Poisson textures, they are independent, and their semi-invariants add (Equation (3)). The symmetrizing operation (A8) thus yields a new texture whose *odd* moments and semi-invariants are zero, and whose *even* moments and semi-invariants are unchanged. For example, application of the operation sym(.) to a texture which is a Poisson superposition of bright discs results in a Poisson superposition of both bright and dark discs, each at half the density of the original texture. Note that application of the symmetrizing operation sym(.) to a texture A and its contrast-reversal $-A$ produces the *same* texture.

We can combine this symmetry principle and the 'brute-force' construction (35) to create iso-$(R^*, 3)$ textures. Let us assume that a texture $\bar{A}$ has been constructed by (35) to be an iso-$(R^*, 2)$ texture corresponding to a given Poisson texture A. The textures sym(A) and sym($\bar{A}$) must be iso-$(R^*, 2)$ textures (since sym(.) leaves even-order moments unchanged). Furthermore, since both symmetrized textures have all *odd* semi-invariants equal to zero, they are iso-$(R^*, 3)$ textures as well.

If two textures are iso-(R^*, p) pairs, they remain iso-(R'^*, p) pairs when the set R^* is augmented by receptive fields which are polarity-reversals of the original set. Thus, in a sense, the symmetrizing operation merely amounts to augmenting the set R^*, and considering superpositions (33) over a larger set.

This construction can be shortened if the only nonzero semi-invariants of A or $\bar{A}$ are of even order. For example, we consider a texture A (Fig. 2A) which consists of a Poisson superposition of profiles $D\mathrm{Gau}_1$. We use a set R^* which consists of $D\mathrm{Gau}_0$ and $D\mathrm{Gau}_2$. Statistically, the texture A is transformed into its contrast-reversal by a reflection. However, since the profiles $D\mathrm{Gau}_0$ and $D\mathrm{Gau}_2$ are left invariant by this reflection, the distribution of responses of $D\mathrm{Gau}_0$ and $D\mathrm{Gau}_2$ to the texture A must be even-symmetric (and have odd-order semi-invariants equal to zero). Hence, the symmetrizing operation need only be applied to the texture $\bar{A}$, since the odd-order semi-invariants of the original texture A are already zero. The resulting iso-$(R^*, 3)$ texture is shown in Fig. 2B.

Application of the symmetrizing operation to the original texture A will result in a third iso-$(R^*, 3)$ texture. This texture, sym(A), is shown in Fig. 2C. There is an unexpected bonus of this construction: the textures of Fig. 2A and C are iso-(R^*, ∞) textures for any receptive field set R^* which contains only even-symmetric profiles. This may be seen as follows. The *even* semi-invariants of A and sym(A) unchanged by the symmetrizing operation (A8), and the *odd* semi-invariants of sym(A) are guaranteed to be zero. Because the texture A is transformed into its contrast-reversal by a rotation of 180 deg, its *odd* semi-invariants with respect to any receptive-field profile left invariant by this rotation must also zero. Thus, *even* semi-invariants of A and sym(A) agree by virtue of general properties of sym(.), and *odd* semi-invariants (for even-symmetric receptive fields) agree because of a special symmetry of the texture A.